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Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping–source interaction

Marcelo M. Cavalcanti [a,*,1], Valéria N. Domingos Cavalcanti [a,2], Irena Lasiecka [b,3]

[a] *Department of Mathematics, State University of Maringá, 87020-900, Maringá, PR, Brazil*

[b] *Department of Mathematics, University of Virginia, Charlottesville, VA 22903, USA*

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Abstract

We establish, subject to some natural additional assumptions imposed on the relation between the source and the damping, both well-posedness and effective optimal decay rates for the solutions of a semilinear model of the wave equation. The theory presented allows to consider both superlinear and sublinear behaviours of the dissipation in the presence of unstructured sources.

1. Introduction

We consider the following model of semilinear wave equation with a nonlinear boundary dissipation and nonlinear boundary/interior sources:

$$\begin{cases} u_{tt} = \Delta u + f(u) \quad \text{in } \Omega \times (0, \infty), \\ \partial_\nu u + u + g(u_t) = h(u) \quad \text{on } \Gamma \times (0, \infty), \\ u(0) = u_0 \in H^1(\Omega), \quad u_t(0) = u_1 \in L_2(\Omega). \end{cases} \tag{1.1}$$

* Corresponding author.

E-mail address: mmcavalcanti@uem.br (M.M. Cavalcanti).

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Here the operators $f(u)$, $g(u_t)$, $h(u)$ are Nemytskii operators associated with scalar, continuous functions $f(s)$, $g(s)$, $h(s)$ defined for $s \in R$. The function $g(s)$ is assumed monotone.

The main goal of this paper is twofold: (i) to study well-posedness of the system given by (1.1) on the finite energy space, i.e. $H^1(\Omega) \times L_2(\Omega)$, and (ii) to derive uniform decay rates of the energy when $t \to \infty$. The well-posedness includes existence and uniqueness of both local and global solutions.

The main difficulty and the novelty of the problem considered is related to the presence of the boundary nonlinear term $h(u)$. This difficulty has to do with the fact that Lopatinski condition does not hold for the Neumann problem. The above translates into the fact that in the absence of the damping, the linear map $h \to (u(t), u_t(t))$, is not bounded from $L_2(\Sigma) \to H^1(\Omega) \times L_2(\Omega)$, unless the dimension of Ω is equal to one. In fact, the maximal amount of regularity that one obtains in general is $H^{2/3}(\Omega) \times H^{-1/3}(\Omega)$ [27,32]. The lack of sufficient regularity is a major predicament in studying nonlinear problems, within the finite energy framework, and with the nonlinearity located on the boundary. Indeed, no matter how smooth or regular nonlinearity $h(u)$ is, the effect of this nonlinearity is not only non-Lipschitz with respect to the phase space but also non-Lipschitz with respect to the weak semigroup formulation of solutions (unlike the Dirichlet problem for which Lopatinski condition is satisfied). This difficulty has been recognized a long time ago and dealt with (particularly in the context of control theory) by exploiting dissipation as a sort of "regularization," [19,23,25]. The main principle behind these works can be formulated as follows "strong boundary damping implies regularity" hence compensates for the roughness of solutions driven by the non-homogeneous Neumann boundary data. In fact, even linear dissipation $g(u_t) = \alpha^2 u_t$ changes the problem to the one where Lopatinski condition is satisfied [25]. Indeed, the boundary damped equation

$$\begin{cases} u_{tt} = \Delta \quad \text{in } \Omega \times (0, \infty), \\ \partial_\nu u + u + \alpha^2 u_t = h \quad \text{on } \Gamma \times (0, \infty), \\ u(0) = 0, \quad u_t(0) = 0 \end{cases} \tag{1.2}$$

has finite energy solutions with L_2 boundary input, i.e. $h \in L_2(\Sigma)$ (which is not the case when $\alpha = 0$). This property has been since used in control theory of PDEs, particularly in the context of stabilization and Riccati Equations [25, and references therein].

Thus, it is clear and it has been recognized a long time ago, that well-posedness theory with semilinear boundary nonlinearity and finite energy solutions must rely, and take advantage of the boundary dissipation (without changing the equation). In fact, this philosophy has been pursued in number of works [17,23] where construction of finite energy solutions to nonlinear problems did rely on "smoothing" effect created by the boundary damping. In [23] it was shown that any dissipation g that is continuous, monotone and bounded linearly from below at infinity produces finite energy solutions in the presence of a boundary source h of a subcritical growth. In [18] more general abstract version of the wave equation has been considered and full well-posedness theory has been derived for dynamics with nonlinear damping–source interaction on the boundary. Here, again, the presence of the damping is the driving force behind local well-posedness of the solutions. Well-posedness for wave equation *without boundary dissipation* and *with* nonlinear boundary sources has been treated in [22] and [26] and also [4]. In the absence of the boundary dissipation, the effect of nonlinear boundary source is offset by sharp regularity of the Neumann problem [27]. Additional gain of fractional derivative on the boundary allows to apply fixed point and prove existence and uniqueness of solutions in spaces with topology *above* the finite energy.

More recently, the issue of well-posedness of solutions to wave equation with semilinear boundary conditions has been resurrected and it has attracted considerable attention, as, for example, [5,6,34] and references therein, particularly with respect to a better understanding of a damping–source interaction located on the boundary. Reference [34] studies homogeneous wave equation ($f = 0$) with both semilinear boundary conditions and the damping of a polynomial structure. By exploiting the regularizing effect of the damping, [34] proves existence of finite energy solutions with boundary functions g, h of a polynomial structure. The study of [34] was followed by [8] where the authors of this latter paper have additionally established exponential decay rates for the problem with *linear* damping and small initial data taken from a potential well. This was accomplished by adapting the method introduced in [29,30].

The present paper is in response to this renewed interest in the Neumann damped problem with semilinearity located on the boundary. The main aim of this manuscript, with principal results stated in Section 2, is to show that far more general existence and stability results can be proved by applying essentially the same technique as in [23] that are based on a natural tool for the problem—monotone operator theory. This is in contrast with the Schauder fixed point arguments used in [34]. We note that the semilinear problem considered in (1.1) *is not monotone*, however the monotonicity of g allows to adapt successfully [23] some aspects of monotone methods. By extending the techniques of [9,23] we establish, subject to some natural additional assumptions imposed on the relation between the source and the damping, both well-posedness and optimal decay rates for the energy function. Sections 3–6 deal with well-posedness issues, while Section 7 describes asymptotic (in time) decay rates enjoyed by solutions to the problem. In contrast with vast majority of papers written on the subject, [14] and references therein, the present paper makes no assumption on the damping at the origin (except for the continuity and monotonicity) and puts no restrictions of a polynomial structure imposed on both nonlinear term and the damping. As a consequence, the theory presented allows to consider both superlinear and sublinear behaviours at the origin of the dissipation in the presence of unstructured sources. Moreover, the general decay estimate obtained in Theorem 2.6 leads (see Section 8) to computable decay rates (algebraic, logarithmic, etc.), that are determined solely by the behaviour of the dissipation at the origin (without making any a priori assumptions on the behavior of dissipation at the origin). This is accomplished by following the method presented in [23], which was the very first paper to establish optimal decay rates obeyed by the energy function, *without any growth assumptions* imposed at the dissipation at the origin. The main trust of that method is to reduce the problem of computations of decay rates for a PDE to solving an appropriate—explicitly given—ODE of monotone type.

The idea of computing decay rates for general class of damping functions without quantified behavior at the origin, has attracted, since then, considerable attention in the literature. In fact, papers [29,30] provide a general method, based on energy inequalities (adapting the strategy initiated in [13]), which leads to computable decay rates for dissipative systems, under some regularity assumptions imposed on the damping. Though the decay rates obtained in these papers are explicit, they are not optimal for many important cases. Instead, the optimal and computable decay rates were later obtained in [1], where more refined weighted energy methods are used. In fact, the decay rates obtained in [1] are in line with the ones obtained in [23], whenever tested on the same class of problems. Though [23] describes the decay rates in terms of solutions to the explicitly given ODE, the form of these decay rates was not explicitly given in [23]. Nevertheless, these asymptotics are determined, as we shall see below, from the explicit algorithm given in [23]. For the convenience of the reader, Section 8 provides step-by-step procedure leading to an effective solution of the relevant ODE that predicts completely asymptotic behavior of nonlinear PDE with both damping and sources satisfying certain necessary and sufficient “stability” properties. The case of “unstable” sources is treated in Section 9, where a finite time blow up phenomenon is exhibited for finite energy solutions.

2. Main results

Our main results are formulated below. In what follows we shall denote $U(t) \equiv (u(t), u_t(t))$ and $H \equiv H^1(\Omega) \times L_2(\Omega)$.

2.1. Well-posedness

Our preliminary result deals with the case where the dissipation is assumed strongly monotone. In that case, one obtains uniqueness of solutions. Later on we shall consider the same problem but without the strong monotonicity assumption.

Theorem 2.1. *We assume that*:

- *g is continuous and strongly monotone, i.e. there exists a constant $m_0 > 0$ such that $(g(s) - g(v))(s - v) \geqslant m_0|s - v|^2$,*
- *f is locally Lipschitz: $H^1(\Omega) \to L_2(\Omega)$,*
- *$\hat{h}(u) \equiv h(u|_\Gamma)$ is locally Lipschitz: $H^1(\Omega) \to L_2(\Gamma)$.*

Then, there exists a local unique solution $U \in C([0, T_M), H)$ and such that $u_t, \nabla u|_\Gamma \in L_2((0, T_M) \times \Gamma)$, where T_M depends on $|U(0)|_H$ and on m_1, where the constant m_1 is such that $g(s)s \geqslant m_1|s|^2$, $|s| \geqslant 1$.

Global version (with global Lipschitz conditions imposed in f and h) of existence–uniqueness result in Theorem 2.1 was proved in [23]. The present formulation allows to consider locally Lipschitz functions and provides a control of T_M depending only on m_1 and not on m_0. This last feature will play a critical role in extending the existence result of Theorem 2.1 to a more general class of sources.

The result formulated next relinquishes the strong monotonicity assumption. In that case the uniqueness of finite energy solutions is lost. In order to state the corresponding result we formulate the following assumption.

Assumption 2.1. We assume that

- g is monotone and continuous. In addition, $g(s)$ satisfies the following growth condition at *infinity* with some $q > 0$

$$m_q|s|^{q+1} \leqslant g(s)s \leqslant M_q|s|^{q+1}, \quad \text{for } |s| \geqslant 1; \tag{2.1}$$

- f is locally Lipschitz: $H^1(\Omega) \to L_2(\Omega)$;
- $\hat{h}(u) \equiv h(u|_\Gamma)$ is locally Lipschitz from $H^{1-\epsilon}(\Omega) \to L_{(q+1)/q}(\Gamma)$ for some small and positive ϵ.

Theorem 2.2. *Under Assumption* 2.1 *there exists a* local in time *weak solution* $U \in C([0,T_M),H)$*, where* T_M *depends on* $|U(0)|_H$ *and on* m_q*. In addition* $u_t|_\Gamma \in L_{q+1}((0,T_M)\times\Gamma)$*,* $\partial_\nu u \in L_{(q+1)/q}((0,T_M)\times\Gamma)$ *and such solution may not be unique. If one additionally assumes the a priori bound for* $|U(t)|_H$*,* $t>0$*; then* $T_M=\infty$ *and weak solutions satisfy the energy identity* (2.3).

Theorem 2.2 is an extension of Theorem 2.1 in [23] where the latter provides the same result with $q=1$. In the special case, when $f=0$,

$$g(s)s=|s|^{q+1}, \quad \text{and} \quad h(s)=|s|^{k-1}s, \quad \text{with } k<r,\ q>\frac{k}{r-k},\ r=\frac{2(n-1)}{n-2}; \tag{2.2}$$

Theorem 1 in [34] provides local existence of finite energy solutions. It is clear that Assumption 2.1 is not only amply satisfied in that case but it allows: (i) for a larger class of unstructured sources f, h and (ii) for a larger class of dissipations g with the growth restrictions imposed only at infinity. Thus, Theorem 2.2 leads to substantially more general result than that one in [34]. The proof of Theorem 2.2 is based on extending monotone operator theory approach used in [23], rather than critical use of compactness via Schauder fixed point theory, as in [34].

More recently, the result of Theorem 2.2 has been extended in [4] by establishing uniqueness along with Hadamard well-posedness of weak solutions, under the assumption that $h\in C^2$ and $\hat{h}$ is locally Lipschitz from $H^1(\Omega)\to L_2(\Gamma)$.

Global solutions can be obtained under some growth conditions imposed on f and h.

Theorem 2.3. *Solutions referred to in Theorem* 2.2 *are* global *and defined for all* $0\leqslant t\leqslant T$ *with an arbitrary* T *provided*

- $|f(s)|\leqslant M|s|$, $|s|\geqslant 1$,
- $|h(s)|\leqslant M|s|^r$, $|s|\geqslant 1$, $r+1\leqslant\frac{2(n-1)}{n-2}$ *and* $r\leqslant\max[q,2\frac{q}{q+1}]$.

Remark 2.1. One could relax the growth condition imposed on f by adding an appropriate damping term in the interior of the equation. However, this will not add any substance to the novelty of the results. The presence of nonlinear source on the boundary is more intricate and delicate to deal with. For this reason we do not consider internal damping which could have the same regularizing effect on solutions allowing larger class of nonlinear functions $f(s)$ to be considered.

The next class of nonlinearities, considered in the context of global solutions, relinquishes the growth conditions imposed in Theorem 2.3 on the functions f and h. There are two ways of doing this. One way is to impose some structural conditions controlling the bound from below for $f(s)s$ and $h(s)s$. This was done in [23] and also [9] in the context of studying attractors. In general, when the "sign" condition is violated, the energy of solutions may become negative and the solutions blow up in a finite time [3,11,12,16,34]. To prevent this from happening one must impose some restrictions on solutions in order to guarantee that all positive time trajectories "live" in a suitably constraint set. This device is well known and is related to potential well theory. In what follows below we shall describe the corresponding class of problems. In order to formulate our results it is convenient to introduce the energy of the system.

$$E(u,v)=E_p(u)+E_k(v);\qquad E_k(v)\equiv\frac{1}{2}|v|^2_{0,\Omega},$$

$$E_p(u)\equiv\frac{1}{2}\big[|\nabla u|^2_{0,\Omega}+|u|^2_{0,\Gamma}\big]-\int\limits_{\Omega}\mathcal{F}(u)(x)\,dx-\int\limits_{\Gamma}\mathcal{H}(u)(x)\,dx,$$

$\mathcal{F}$ (respectively $\mathcal{H}$) denotes the antiderivative of f (respectively h) and $|u|_{s,D}$ denotes the Sobolev norm $|u|_{H^s(D)}$, where D will be used for Ω or Γ. We have the following energy relation satisfied for weak local solutions (see Theorem 2.2)

$$E\big(u(t),u_t(t)\big)+\int\limits_0^t\int\limits_{\Gamma}g(u_t)u_t\,dx\,dt=E\big(u(0),u_t(0)\big).\tag{2.3}$$

This relation suggests an obvious a priori bound for solutions, assuming that the potential energy is nonnegative and dominated in "some" sense by a "linear" part of the energy. And, in fact, this does happen if the nonlinear functions are of special form and the initial data are taken from a special set—so-called potential well. In the case when one source (be it either boundary or interior) is active and the damping is polynomial, the potential well theory has been developed in [33,34] and references therein. We shall show that a similar construction can be performed for two competing sources (boundary and interior), and without assuming any (polynomial) structure on the damping. To accomplish this, we formulate the following assumption.

Assumption 2.2. We shall assume that functions h and f are of a polynomial structure. That is,

$$f(s)=|s|^{p-1}s,\qquad h(s)=|s|^{k-1}s\tag{2.4}$$

where $p,k\geqslant 1$ are such that $H^1(\Omega)\subset L_{p+1}(\Omega)$, $H^{1/2}(\Gamma)\subset L_{k+1}(\Gamma)$.

Our goal is to determine a set of initial data that is invariant with respect to the flow. To achieve this we introduce the following notation:

- $B_\Omega\equiv\sup_{u\in H^1(\Omega)}\frac{|u|_{L_{p+1}(\Omega)}}{\sqrt{|\nabla u|^2_{0,\Omega}+|u|^2_{0,\Gamma}}}$, $B_\Gamma\equiv\sup_{u\in H^1(\Omega)}\frac{|u|_{L_{k+1}(\Gamma)}}{\sqrt{|\nabla u|^2_{0,\Omega}+|u|^2_{0,\Gamma}}}$,
- λ_0 is the first positive zero of the function $F'(x)$ where

$$F(x)\equiv\frac{1}{2}x^2-\frac{1}{p+1}B_\Omega^{p+1}x^{p+1}-\frac{1}{k+1}B_\Gamma^{k+1}x^{k+1},$$

- $d\equiv F(\lambda_0)$.

Considering the above notation we have the following result.

Theorem 2.4. *Under Assumptions* 2.2 *and* 2.1 *the set*

$$A\equiv\big\{(u_0,u_1)\in H^1(\Omega)\times L_2(\Omega);\ |\nabla u_0|^2_{0,\Omega}+|u_0|^2_{0,\Gamma}\leqslant\lambda_0^2,\ E(0)<d\big\}$$

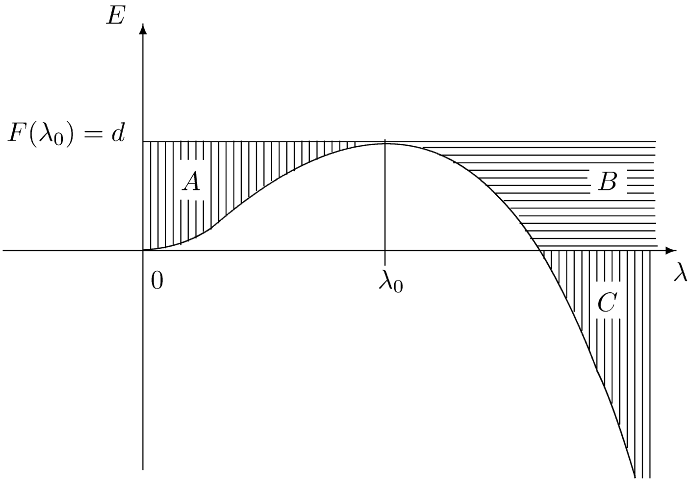

Fig. 1. The regions A and $B \cup C$ are related, respectively, with the existence of global solutions and blow up phenomenon in finite time.

is invariant under the flow. Moreover, solutions corresponding to the initial data $(u_0, u_1) \in A$ *are global. We have that* $T(t)A \subset A$ *and* $|u(t)|_{1,\Omega} \leqslant CE_p(u(t)) \leqslant Cd$, *where* $T(t)$ *denotes solution operator that can be multivalued.*

Before announcing our next result, let us introduce the following set

$$B \cup C = \left\{(u_0, u_1) \in H^1(\Omega) \times L_2(\Omega);\ |\nabla u_0|^2_{0,\Omega} + |u_0|^2_{0,\Gamma} > \lambda_0^2;\ E(0) < d\right\},$$

according to Fig. 1.

Theorem 2.5. *Suppose that Assumptions* 2.1 *and* 2.2 *hold, and, in addition, that* $p, k > 1$. *Moreover, assume that* $(u_0, u_1) \in B \cup C$, *and*

$$k > q \quad (\textit{the boundary source domains the damping}). \tag{2.5}$$

Let u be a weak solution that exists on the interval $[0, T_M)$, *according to Theorem* 2.2.
Assume that one of the following assumptions holds:

(i) $E(0) < 0$,
(ii) $E(0) \geqslant 0$, $E(0) < \frac{\lambda_0^2(k-1)}{2(k+1)}$ *if* $p > k$ *or* $E(0) < \frac{\lambda_0^2(p-1)}{2(p+1)}$ *if* $k > p$,
(iii) $E(0) \geqslant 0$, $E(0) \geqslant \frac{\lambda_0^2(k-1)}{2(k+1)}$ *if* $p > k$ *and the difference* $p - k$ *is small enough or* $E(0) \geqslant \frac{\lambda_0^2(p-1)}{2(p+1)}$ *if* $k > p$ *and the difference* $k - p$ *is small enough.*

Then, $T_M < +\infty$ *and* $\|u(t)\|^{p+1}_{p+1} + \|u(t)\|^{k+1}_{k+1,\Gamma} \to +\infty$ (*and, consequently,* $\|u(t)\|^{p+1}_{\infty,\Omega} + \|u(t)\|^{k+1}_{\infty,\Gamma} \to +\infty$) *as* $t \to T_M^-$.

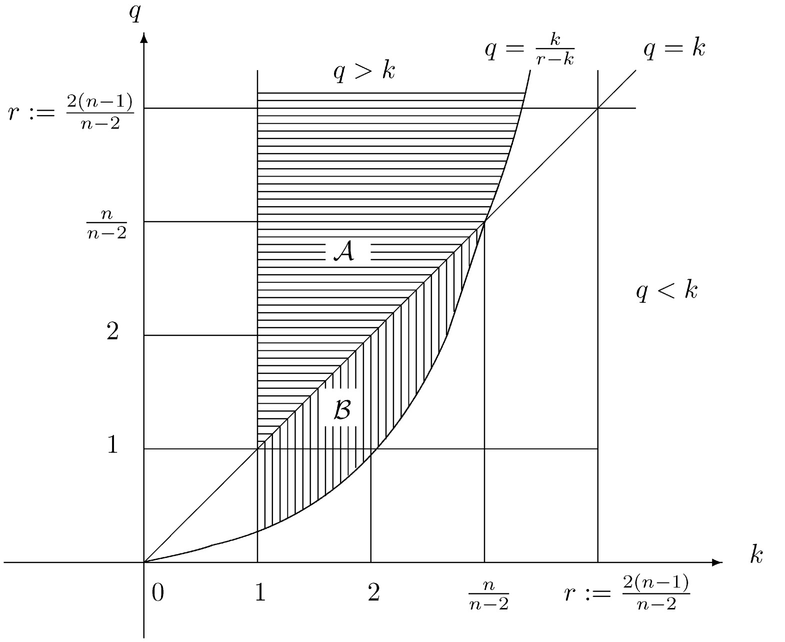

Fig. 2. The regions $\mathcal{A}$ and $\mathcal{B}$ are related, respectively, with the sets of the (k, q) couples regarding to the existence of global solutions and blow up phenomenon in finite time.

Remark 2.2.

- It is important to observe that if $p > k$ then $\frac{\lambda_0^2(k-1)}{2(k+1)} < d$ and if $k > p$ then $\frac{\lambda_0^2(p-1)}{2(p+1)} < d$.
- We are assuming in Theorem 2.5 that the condition (2.1) holds for all $s \in \mathbb{R}$, that is,

$$m_q|s|^q \leqslant \left|g(s)\right| \leqslant M_q|s|^q, \quad \text{for all } s \in \mathbb{R}.$$

- When one considers positive values of the initial energy $E(0)$ greater or equal than the critical number $\frac{\lambda_0^2(k-1)}{2(k+1)}$ if $p > k$ or $\frac{\lambda_0^2(p-1)}{2(p+1)}$ if $k > p$, we are forced to consider the difference $p - k$ or $k - p$ small enough. This is required because of the 'competition' between the two sources which are acting at the same time on the system.

Figure 2 illustrates the set of couples (k, q) related with global existence and blow up phenomenon.

2.2. Uniform decay rates

Having established global well-posedness of solution to (1.1), we focus our attention on decay rates that can be obtained for the energy function. To accomplish this, we need to impose more specific conditions assumed about the sources and the dissipation. In fact, these conditions (in particular linear bound at infinity imposed on the dissipation) are typical in *boundary* stabilization models [14,15,23]. Our standing assumption is the following

Assumption 2.3. Assume that

(1) Function $g \in C(R)$ is monotone, $g(0)=0$, and

$$m_1 s^2 \leqslant g(s)s \leqslant M_1 s^2, \quad \text{for } |s| \geqslant 1.$$

(2) $f \in C^1(R)$, $f(0)=0$ and

$$\left|f'(s)\right| \leqslant C\left[1+|s|^{k_0-1}\right], \quad \text{for all } s \in \mathbb{R},\ 1 \leqslant k_0 \leqslant \frac{n}{n-2},\ n>2 \text{ and } 1 \leqslant k_0 < \infty,\ n=2.$$

(3) $h \in C^1(R)$, $h(0)=0$ and

$$\left|h'(s)\right| \leqslant C\left[1+|s|^{k_1-1}\right], \quad \text{for all } s \in \mathbb{R},\ 1 \leqslant k_1 < \frac{n-1}{n-2},\ n>2 \text{ and } 1 \leqslant k_1 < \infty,\ n=2.$$

We note that the first three parts of Assumption 2.3 imply the hypotheses of Assumption 2.1 (with $q=1$) which, in turn, guarantee local existence of finite energy solutions. In fact, hypotheses made in Assumption 2.3 imply stronger regularity of traces of solutions which imply the boundedness of $u_t, \nabla|_\Gamma u$ on the boundary (see Theorems 2.1 and 2.2 applied with $q=1$).

In order to be able to discuss stability of solutions, we need to work with global solutions. The assumption that guarantees global existence is the following

$$\left|u(t)\right|_{1,\Omega}^2 \leqslant C_1 E_p\big(u(t)\big), \quad t \geqslant 0.$$

Indeed, in view of (2.3) this assumption guarantees an a priori bound. Thus, Assumption 2.3 along with the above mentioned condition, implies that solutions under consideration are uniformly bounded (in time) in finite energy space $H^1(\Omega) \times L_2(\Omega)$. Precise statement regarding regularity of these solutions is given in Lemma 7.1. We note that such solutions, however, may not be unique. The main question we want to address is uniform stability and uniform decay rates for such solutions. Our main result reads as follows:

Theorem 2.6 *(Uniform Decay Rates).*

- *Let Assumption* 2.3 *holds true. Then, there exists a finite energy solution such that* $u \in C([0,T_M), H^1(\Omega)) \cap C^1([0,T_M), L_2(\Omega))$ *and* $u_t, \nabla|_\Gamma u \in L_2(0,T_M; L_2(\Gamma))$.
- *We consider all weak solutions of finite energy such that the regularity listed above holds and, in addition,*

$$\left|u(t)\right|_{1,\Omega}^2 \leqslant C_1 E_p\big(u(t)\big), \quad t \geqslant 0. \tag{2.6}$$

  Then, the corresponding solutions are global and defined on $[0,\infty)$.
- *Moreover, assume that the only stationary solution of* (1.1) *within the class of solutions under consideration is the trivial solution. Under the above assumptions these solutions satisfy the decay estimate driven by a real variable positive function* $S(t)$ *defined as the solution of the differential equation*

$$S_t + q^*(S) = 0, \qquad S(0) = E(0) = S_0, \tag{2.7}$$

where $q^*(S)$ *is a continuous, monotone increasing function given by* (7.22). *We have that* $E(t) \leqslant S(\frac{t}{T_0} - 1)$ *for* $t > T_0$ *and the function* $S(t)$ *decays uniformly to zero.*

Remark 2.3. Decay rates for weak solutions are described by solutions of nonlinear, dissipative ODE given in (2.7) (as in [23]). The quantity defining this ODE—monotone increasing function q^* given in (7.21)—depends only on the behaviour of the dissipation g at the origin and constants that are intrinsic to the model. This allows for explicit calculation of the decay rates—as shown in Section 8. In order to facilitate resolution of the ODE, one may approximate given functions by functions that have the same "comparable" behaviour at the origin. By doing this and, relying on a comparison theorem for dissipative ODE, one may obtain a simpler ODE to solve, that predicts the exact decay rates. This procedure is illustrated in Section 8.

Remark 2.4. Note that the theorem above applies to all weak solutions which are in a potential well and are described by Theorem 2.4. In addition the result applies to all other nonlinear problems that lead to dissipative dynamics (like in [9,23]). Thus, the present formulation encompasses all cases of interest without imposing a priori unnecessary restrictions on the structure of nonlinearities involved. This level of generality was achieved by using an "intrinsic" stability method developed in [23].

Remark 2.5. Note that under the assumptions imposed, weak solutions may not be necessarily unique. Nevertheless, the decay rates announced in Theorem 2.6 are valid for all weak solutions with the prescribed properties. This is achieved by employing a rather special approximation argument developed in [23].

Few words about references and literature related to uniform stability and decay rates associated with a nonlinear wave equation and boundary damping. The very first result concerning uniform decay rates without imposing any growth conditions at the origin on the damping function $g(s)$ was given in [23]. This reference also treats simultaneously boundary and interior sources. Theorem 2.6, though similar in the spirit to [23], extends significantly [23] by allowing a much larger class of sources to be considered. Indeed, the only condition that is assumed in Theorem 2.6 is condition (2.6) which can be interpreted as Liapunov type of stability. This requirement (2.6) is satisfied for: (i) sources with appropriate dissipative bounds at infinity [9,23], and also for (ii) sources corresponding to the potential well theory. Clearly, condition (2.6) is necessary for any sort of stability (hence for the decay rates). A sharp answer to the question of how large is a class of sources that complies with this requirement, is still unknown. A relevant stability analysis in the case of *interior damping and interior sources* is given in [10]. In the special case when $f = 0$, the damping is linear, that is, $g(s) = \text{const} \cdot s$ and the boundary source is of the polynomial structure $h(s) = |s|^{k-1}s$, $k < \frac{2(n-1)}{n-2}$, exponential decay rates for small data taken from potential well are proved in [8]. The case of sole interior source $f(s) = |s|^{k-1}k$ with subcritical exponent $k < \frac{n}{n-2}$, $h = 0$ and nonlinear boundary damping $g(s)$ was treated in [7]. In [7] uniform decay rates similar to these obtained earlier in [29] were derived for small initial data taken from potential well constructed in [33]. Thus the results of Theorem 2.6, which does not assume any a priori structure of the sources and allows for a completely arbitrary behaviour of the damping at the origin, subsume and significantly extend the results obtained in the prior literature.

A purely dissipative case of the wave equation, i.e., $f = 0$, $h = 0$ and subject to smooth, nonlinear, monotone unstructured at the origin damping $g(s)$, has been considered in the recent paper [1]. In fact, [1] provides a very complete and optimal treatment of the decay rates obtained for this class of problems. As we shall see in Section 8, our general Theorem 2.6, when specialized to classes of problems considered in [1], provides exact and computable decay rates which are quantitatively the same as in [1] but obtained for a larger class of models that include "non-smooth" dissipation and the presence of boundary–interior sources. One should also note that the method used in [1] and based on weighted integral inequalities is very different from that used in the present paper, where the latter is essentially an extension of the technique introduced in [23]. More specifically, the weighted energy method developed in [1] is close in a spirit to Liapunov's function methods introduced in [13] and extended later in [14,29]. However, the use of the special weights built into Liapunov's functions in [1] pushes the Liapunov's based analysis much further and leads in [1] to sharp decay rates with precise constants. Instead, the technique introduced in [23], and followed in this paper, is not Liapunov's function based. It relies on a "finite time interval" PDE analysis which is then propagated into an ordinary differential equation (Eq. (2.7) in Theorem 2.6). The key to this method is a construction of a special concave function (see (7.19)) that captures the behaviour of the dissipation (a priori unstructured) at the low frequency range. This function determines the decay rates which are just read-off from the corresponding ODE. To our best knowledge, F. Alabau-Boussouira in [1], was the first who gave an explicit expression to the function appearing in Section 8, namely function $(h^*)^{-1}$. The resulting decay rates are optimal and applicable to a large variety of models. The method is flexible, insensitive to compact perturbations (as long as the overall system is just strongly stable—a minimal a priori requirement) and works well with unstructured nonlinearities, nondifferentiable dissipation and equations that may not have unique solutions. On the other hand, the pay-back for the flexibility of the method is lesser control, when compared with [1], of the specific constants appearing in the estimates (see Remark 2.3).

The proofs of theorems are given in Sections 3–7. Section 8 is devoted to a general algorithm allowing for precise calculations of the decay rates announced in Theorem 2.6.

3. Proof of Theorem 2.1

As mentioned before, the main difficulty of the problem under study is the fact that the Neumann problem does not satisfy Lopatinski condition and therefore, the map from the boundary data in $L_2(\Sigma)$ into finite energy space H is not bounded (unless dimension of Ω is equal to one). In order to cope with the problem, we introduce a regularizing term—strongly monotone dissipation, whose effect is to 'force' the Lopatinski condition. The lemma below reflects this property and is the first step in the proof.

Step 1. Globally Lipschitz functions f and h

Lemma 3.1. *Assume that*

Assumption 3.1.

- functions h and f are globally Lipschitz on $\mathbb{R}$, with Lipschitz constants L and M, respectively,
- g is continuous and strongly monotone, that is, there exists $m_0 > 0$ such that $(g(s) - g(t))(s-t) \geqslant m_0|s-t|^2$,
- $\hat{h}$ is Lipschitz: $H^1(\Omega) \to L_2(\Gamma)$, that is, $|\hat{h}(u) - \hat{h}(v)|_{0,\Gamma} \leqslant L_0|u-v|_{1,\Omega}$,

- $\hat{f}$ is Lipschitz $H^1(\Omega) \to L_2(\Omega)$, that is, $|\hat{f}(u) - \hat{f}(v)|_{0,\Omega} \leqslant L_1|u - v|_{1,\Omega}$.

Then, Eq. (1.1) *has a unique, global in time solution* $u \in C([0,T]; H^1(\Omega)) \cap C^1([0,T]; L_2(\Omega))$, *where* T *is arbitrary.*

Proof. The proof of this result is given in [23]. For reader's convenience we repeat the steps. As usual we introduce the operator $A : D(A) \subset H \to H$ given by

$$A\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ \Delta_N(u - N\hat{h}(u) + N\hat{g}(v)) + \hat{f}(u) \end{bmatrix}$$

where Δ_N corresponds to Laplacian with Neumann–Robin boundary conditions, namely,

$$\Delta_N u = \Delta u \quad \text{with } D(\Delta_N) = \{u \in H^2(\Omega);\ \partial_\nu u + u = 0 \text{ on } \Gamma\}.$$

The operator $N : H^s(\Gamma) \to H^{s+3/2}(\Omega)$, $s \in \mathbb{R}$, is the usual Neumann map which is defined as a harmonic function driven by the Neumann boundary data, we say,

$$N\varphi = w \quad \Leftrightarrow \quad \begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \partial_\nu w + w = \varphi & \text{on } \Gamma. \end{cases}$$

The domain of A is given by

$$D(A) = \{(u, v) \in (H^1(\Omega))^2;\ u - N\hat{h}(u) + N\hat{g}(v) \in D(\Delta_N)\}.$$

Due to the monotonicity of g, the domain $D(A)$ is dense in H. Indeed, it suffices to take u sufficiently smooth and build in boundary conditions by solving $g(v) = -\partial_\nu u - u + \hat{h}(u)$ on the boundary. Then, (1.1) can be written as

$$\partial_t \begin{bmatrix} u \\ u_t \end{bmatrix} = A\begin{bmatrix} u \\ u_t \end{bmatrix}.$$

This framework is well known and goes back to [19,20]. Comprehensive treatment of semigroup boundary PDE models is given in [25].

In order to prove the lemma it suffices to show that the operator $A - \omega I$ is m-dissipative on H for some positive ω. We endow H with a standard inner product $D((-\Delta_N)^{1/2}) \times L_2(\Omega)$, that is

$$\big((u, v), (\hat{u}, \hat{v})\big)_H = \big((-\Delta_N)^{\frac{1}{2}} u, (-\Delta_N)^{\frac{1}{2}} \hat{u}\big)_{0,\Omega} + (v, \hat{v})_{0,\Omega}.$$

Dissipativity

Proof of dissipativity is standard with the exception of the following line of computations which reflects the presence of the boundary source and the benefit of strong monotonicity of the dissipation. Let $U = (u, v) \in D(A)$, $\hat{U} = (\hat{u}, \hat{v}) \in D(A)$.

$$
\begin{aligned}
&\big(A(U)-A(\hat U),U-\hat U\big)_H-\omega|U-\hat U|_H^2\\
&\quad\leqslant \big((-\Delta_N)^{\frac12}(v-\hat v),(-\Delta_N)^{\frac12}(u-\hat u)_{0,\Omega}\\
&\qquad+\big(\Delta_N\big\{(u-\hat u)-N\big[\hat h(u)-\hat h(\hat u)\big]+N\big[\hat g(v)-\hat g(\hat v)\big]\big\},\ v-\hat v\big)_{0,\Omega}\\
&\qquad+\big(\hat f(u)-\hat f(\hat u),v-\hat v\big)_{0,\Omega}-\omega\big[|u-\hat u|_{1,\Omega}^2+|v-\hat v|_{0,\Omega}^2\big]\\
&\quad\leqslant\big\langle\hat h(u)-\hat h(\hat u),v-\hat v\big\rangle_\Gamma-\big\langle\hat g(v)-\hat g(\hat v),v-\hat v\big\rangle_\Gamma\\
&\qquad-\omega\big[|u-\hat u|_{1,\Omega}^2+|v-\hat v|_{0,\Omega}^2\big]+L_1|u-\hat u|_{1,\Omega}|v-\hat v|_{0,\Omega}\\
&\quad\leqslant -m_0|v-\hat v|_\Gamma^2+L_0|u-\hat u|_{1,\Omega}|v-\hat v|_\Gamma-\left(\omega-\frac{L_1}{2}\right)\big[|u-\hat u|_{1,\Omega}^2+|v-\hat v|_{0,\Omega}^2\big]\\
&\quad\leqslant -(m_0-\epsilon)|v-\hat v|_\Gamma^2+\left(L_0^2C_\epsilon+\frac{L_1}{2}-\omega\right)|u-\hat u|_{1,\Omega}^2+\left(\frac{L_1}{2}-\omega\right)|v-\hat v|_{0,\Omega}\leqslant 0,
\end{aligned}
$$

where the last inequality is concluded for ϵ smaller than m_0 and ω larger than $L_0^2C_\epsilon+\frac{L_1}{2}$.

Maximal dissipativity

We need to show that the operator $A-\omega I-\lambda I$ is onto $H=H^1(\Omega)\times L_2(\Omega)$ for some $\lambda>0$. We can consider, without loss of generality, $\omega=0$, otherwise we adjust λ. This amounts to solvability for $(u,v)\in D(A)$ of the following system of equations

$$
\begin{cases} v-\lambda u=a,\\ \Delta_N[u-N\hat h(u)+N\hat g(v)]+\hat f(u)-\lambda v=b, \end{cases}\tag{3.1}
$$

for given $(a,b)\in H^1(\Omega)\times L_2(\Omega)$.

This is equivalent to

$$
\frac{1}{\lambda}\Delta_N v-\Delta_N N\hat h\left(\frac{v-a}{\lambda}\right)+\Delta_N N\hat g(v)+\hat f\left(\frac{v-a}{\lambda}\right)-\lambda v=b+\frac{1}{\lambda}\Delta_N a\equiv\hat b\in V'
$$

where $V=D((-\Delta_N)^{1/2})\sim H^1(\Omega)$ and Δ_N denotes the isometric extension of Δ_N to the space V into V'.

Thus, the issue reduces in proving surjectivity of the operator

$$
Tv\equiv\frac{1}{\lambda}\Delta_N v-\Delta_N N\hat h\left(\frac{v-a}{\lambda}\right)+\Delta_N N\hat g(v)+\hat f\left(\frac{v-a}{\lambda}\right)-\lambda v
$$

from V onto V'.

Let $Tv=Av+Bv$, $v\in V$, where

$$
Av=\Delta_N N\left[-\hat h\left(\frac{v-a}{\lambda}\right)+\hat g(v)\right]\quad\text{and}\quad Bv=\frac{1}{\lambda}\Delta_N v+\hat f\left(\frac{v-a}{\lambda}\right)-\lambda v.
$$

Since we are working with V, V' framework and $\Delta_N N: L_2(\Gamma)\to V'$ is bounded, the operator

$$
-\Delta_N N\hat h\left(\frac{v-a}{\lambda}\right)\quad\text{is Lipschitz } V\to V'
$$

with a Lipschitz constant proportional to $\frac{1}{\lambda}L$. On the other hand, $\Delta_N N\hat{g}(v)$ is maximal monotone $V \to V'$ for $\lambda \geqslant \frac{L}{m_0}$. Hence $-A$ is monotone $V \to V'$ for large λ. In addition, $-A$ is hemicontinuous and, consequently, $-A$ is maximal monotone [2, Theorem 1.3, p. 40]. Analogously, we obtain that $-B$ is monotone and continuous $V \to V'$, then $-B$ is maximal monotone. Moreover, $-A-B$ is coercive which implies that $-T$ is monotone, hemicontinuous and coercive $V \to V'$. By the surjectivity theorem [2, Theorem 1.3, p. 40] $-T$ is maximal monotone and is also surjective. So, we have proved that T is surjective, that is, we have proved the existence of v in $V = H^1(\Omega)$ and $u = \frac{v-a}{\lambda} \in V$. One easily shows that the pair (u, v) is also in $D(\mathcal{A})$. Indeed, from (3.1) we have

$$\Delta_N\big(u - N\hat{h}(u) + N\hat{g}(v)\big) = \lambda v - \hat{f}(u) + b \in L_2(\Omega),$$

hence $(u - N\hat{h}(u) + N\hat{g}(v)) \in D(\Delta_N)$ as desired.

The proof of maximal dissipativity is thus completed. From nonlinear semigroup theory we obtain unique existence of the solution $U \in C([0,T]; H)$, for any finite $T > 0$ [31, Corollary 4.1, p. 181]. This completes the proof of Lemma 3.1. □

Next step toward the proof of Theorem 2.1 is to relinquish this requirement of globality. We proceed here as in [23] or [9].

Step 2. Truncation of locally Lipschitz functions f and h

As in [9] we truncate functions f and h to obtain globally Lipschitz functions f_K, h_K. This is done by defining

$$f_K(u) \equiv \begin{pmatrix} f(u); & |u|_{1,\Omega} \leqslant K \\ f(\frac{Ku}{|u|_{1,\Omega}}), & |u|_{1,\Omega} \geqslant K \end{pmatrix},$$

$$h_K(u) \equiv \begin{pmatrix} \hat{h}(u), & |u|_{1,\Omega} \leqslant K \\ \hat{h}(\frac{Ku}{|u|_{1,\Omega}}), & |u|_{1,\Omega} \geqslant K \end{pmatrix},$$

where $|u|_{1,\Omega}$ denotes the H^1 norm of u, that is, $|u|^2_{1,\Omega} = |u|^2_{0,\Omega} + |\nabla u|^2_{0,\Omega}$ or, equivalently, $|u|^2_{1,\Omega} = |u|^2_{0,\Gamma} + |\nabla u|^2_{0,\Omega}$.

With the truncated f_K, h_K we consider the following K problem:

$$\begin{cases} u_{tt} = \Delta u + f_K(u) & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u + u + g(u_t) = h_K(u) & \text{on } \Gamma \times (0,\infty), \\ u(0) = u_0 \in H^1(\Omega), \quad u_t(0) = u_1 \in L_2(\Omega), \end{cases} \tag{3.2}$$

where g, f and h satisfy Assumption 2.3.

As shown in [9], for each value of K, f_K, h_K are globally Lipschitz with Lipschitz constants bounded by $L_f(K)$, $L_h(K)$, respectively. Let $L(K)$ denotes the maximum of the two constants.

By the previous Lemma 3.1, problem (3.2) has a unique global solution $U_K(\cdot) \in C([0,T]; H)$ for any finite $T > 0$. In addition, the following estimate holds for strong solutions (obtained from monotone operator theory).

$$\begin{aligned}&\left|u(t)\right|_{1,\Omega}^2+\left|u_t(t)\right|_{0,\Omega}^2+2\int_0^t\int_\Gamma g(u_t)u_t\,dx\,ds\\&\quad\leqslant\left|u(0)\right|_{1,\Omega}^2+\left|u_t(0)\right|_{0,\Omega}^2+2L(K)\int_0^t\left|u(s)\right|_{1,\Omega}\left|u_t(s)\right|_{0,\Gamma}\,ds\\&\qquad+2L(K)\int_0^t\left|u(s)\right|_{1,\Omega}\left|u_t(s)\right|_{0,\Omega}\,ds.\end{aligned}\tag{3.3}$$

By exploiting the strong monotonicity of g and the fact that there exists $m_1>0$ such that

$$g(s)s\geqslant m_1s^2,\quad |s|\geqslant 1,\tag{3.4}$$

so that for all $t>0$, for all $\epsilon>0$ and for some $C_g>0$,

$$\begin{aligned}&\left|u(t)\right|_{1,\Omega}^2+\left|u_t(t)\right|_{0,\Omega}^2+2m_1\int_0^t\int_\Gamma |u_t|^2\,dx\,ds-C_gt-m_1\operatorname{meas}(\Gamma)T\\&\quad\leqslant\left|u(0)\right|_{1,\Omega}^2+\left|u_t(0)\right|_{0,\Omega}^2+\epsilon\int_0^t|u_t|_{0,\Gamma}^2\,ds\\&\qquad+L(K)\left(\frac{L(K)}{\epsilon}+1\right)\left[\int_0^t\left|u(s)\right|_{1,\Omega}^2\,ds+\int_0^t\left|u_t(s)\right|_{0,\Omega}^2\,ds\right].\end{aligned}\tag{3.5}$$

Choosing $\epsilon=m_1$ and applying Gronwall's inequality we have for all $t<T$,

$$\begin{aligned}&\left|u(t)\right|_{1,\Omega}^2+\left|u_t(t)\right|_{0,\Omega}^2+m_1\int_0^t\int_\Gamma |u_t|^2\,dx\,ds\\&\quad\leqslant\left[\left|u(0)\right|_{1,\Omega}^2+\left|u_t(0)\right|_{0,\Omega}^2+C_gT\right]e^{L(K)(1+\frac{L(K)}{m_1})t}.\end{aligned}\tag{3.6}$$

The estimate above is obtained first for strong solutions and then extended to all weak (finite energy) solutions by standard weak lower semicontinuity argument. We note that the stronger estimate in (3.3) (with the inclusion of the damping g) may not hold for weak solutions unless some additional regularity hypotheses are imposed on function $g(s)$.

Now, we notice that taking K large enough so that $|u(0)|_{1,\Omega}^2+|u_t(0)|_{0,\Omega}^2+C_gT<K$ there exists

$$T_K=\min\left\{T,\frac{1}{L(K)(1+\frac{L(K)}{m_1})}\log\frac{K}{|u(0)|_{1,\Omega}^2+|u_t(0)|_{0,\Omega}^2+C_gT}\right\}$$

such that for all $t<T_K$ we have $|u(t)|_{1,\Omega}^2\leqslant K$.

Thus for $t < T_K$, $f_K(u) = f(u)$ and $h_K(u) = h(u)$ in Eq. (3.2). Because of the uniqueness of weak solutions (recall that g is strongly monotone) the solution to the truncated problem coincides with the solution to the original, untruncated equation on that interval.

By reiterating the procedure (with a larger K) we obtain maximal time of the existence T_M, which does depend on m_1 (but it does not depend on m_0). Thus, we have proved local existence of solutions for the problem under the assumption that g is strictly monotone, coercive at infinity, and functions f and h are locally Lipschitz. The proof of existence of solutions in Theorem 2.1 is complete. The additional regularity on the boundary follows from (3.6), boundary conditions in (1.1) and the argument in [28] that establishes the L_2 regularity of the tangential component of the trace of a solution (this argument parallels hidden regularity estimates given in [21]).

For future reference we shall provide below a more general version of the bound in (3.6) that does not involve the constant m_1 but a more general constant m_q, for some $q > 0$, that corresponds to coercivity condition of the order q. More specifically, we have the following corollary:

Corollary 3.2. *Along with the assumptions of Theorem* 2.1 *we assume that for some* $q > 0$ *and for each* $R > 0$ *there exists* $L_q(R)$ *verifying the following estimates*:

$$g(s)s \geqslant m_q s^{q+1}, \quad |s| \geqslant 1, \tag{3.7}$$

$$\left|\hat{h}u - \hat{h}(v)\right|_{L_{\frac{q+1}{q}}(\Gamma)} \leqslant L_q(R)|u - v|_{1,\Omega}, \quad \text{for } |u|_{1,\Omega}, |v|_{1.\Omega} \leqslant R. \tag{3.8}$$

Then for every initial condition satisfying $|U(0)|_H \leqslant R$, *there exists* $K > 0$ *such that the solution verifies* $|U(t)|_H \leqslant K$, *for* $t \in [0, T_M]$, *where the time* T_M *of survival of solutions in Theorem* 2.1 *depends only on* m_q, $L_q(K)$, $L_f(K)$ *and* R.

Remark 3.1. Notice that the original condition (3.4) is satisfied with $q = 1$, thus it is a special case of (3.7). The point we want to make is that under the condition that $\hat{h}$ is locally Lipschitz from $H^1(\Omega) \to L_{(q+1)/q}(\Gamma)$, i.e. condition (3.8) is in force, the survival time of solutions depends only on m_q and the Lipschitz constants (functions) L_q and not necessarily on m_1 and Lipschitz constants L_2. The above property will allow us later to extend the existence part of Theorem 2.1 to a larger class of nonlinear functions h and g.

Proof. The proof of the corollary is based on similar computations as those leading to (3.6) with an appropriate use of Holder's inequality. Indeed, by using at this time condition (3.7) along with Young's inequality, we obtain, for each value of $K > 0$,

$$\begin{aligned}
&\left|u(t)\right|_{1,\Omega}^2 + \left|u_t(t)\right|_{0,\Omega}^2 + 2m_q \int_0^t \int_\Gamma |u_t|^{q+1}\, dx\, ds - C_g t \\
&\quad \leqslant \left|u(0)\right|_{1,\Omega}^2 + \left|u_t(0)\right|_{0,\Omega}^2 + \epsilon \int_0^t \int_\Gamma |u_t|^{q+1}\, dx\, ds \\
&\qquad + L_f^2(K)\left[\int_0^t \left|u(s)\right|_{1,\Omega}^2 ds + \int_0^t \left|u_t(s)\right|_{0,\Omega}^2 ds\right] + C_\epsilon \int_0^t \int_\Gamma \left|h_K(u)\right|^{\frac{q+1}{q}}\, dx\, ds.
\end{aligned} \tag{3.9}$$

From Lipschitz condition in (3.8) we easily obtain for any $R > 0$ that

$$\left|\hat{h}(u)\right|_{L_{\frac{q+1}{q}}(\Gamma)} \leqslant \left|\hat{h}(0)\right|_{L_{\frac{q+1}{q}}(\Gamma)} + L_q(R)|u|_{1,\Omega}, \tag{3.10}$$

provided $|u|_{1,\Omega} \leqslant R$.

As we are interested in $K > 0$ sufficiently large, we are going to assume, without loss of generality, $K \geqslant 1$. Let us consider, firstly, $q \geqslant 1$ which implies that $\frac{q+1}{q} \leqslant 2$.

If $|u|_{1,\Omega} \leqslant K$ and $|u|_{1,\Omega} \geqslant 1$, we have $h_K(u) = \hat{h}(u)$ and then, from (3.10)

$$|h_K u|^{\frac{q+1}{q}}_{L_{\frac{q+1}{q}}(\Gamma)} \leqslant \left\{\left|\hat{h}(0)\right|_{L_{\frac{q+1}{q}}(\Gamma)} + L_q(K)|u|_{1,\Omega}\right\}^{\frac{q+1}{q}} \leqslant C_{h,q} + C_q L_q(K)^{\frac{q+1}{q}} |u|^2_{1,\Omega}. \tag{3.11}$$

If $|u|_{1,\Omega} \leqslant 1$, (3.10) implies $|h_K u|^{(q+1)/q}_{L_{(q+1)/q}(\Gamma)} \leqslant C_{h,q} + C_q L_q(1)^{(q+1)/q}$.

Defining $C_h = C_{h,q} (= C_q|\hat{h}(0)|^{(q+1)/q}_{L_{(q+1)/q}(\Gamma)}) + C_q L_q(1)^{(q+1)/q}$ we have for $|u|_{1,\Omega} \leqslant K$ that

$$|h_K u|^{\frac{q+1}{q}}_{L_{\frac{q+1}{q}}(\Gamma)} \leqslant C_h + C_q L_q(K)^{\frac{q+1}{q}} |u|^2_{1,\Omega}. \tag{3.12}$$

Considering the case $|u|_{1,\Omega} \geqslant K$ and, consequently, $h_K(u) = \hat{h}(\frac{Ku}{|u|_{1,\Omega}})$ we get

$$\int_\Gamma \left|h_K(u)\right|^{\frac{q+1}{q}} dx \leqslant C_h + C_q L_q^{\frac{q+1}{q}}(K) K^{\frac{1-q}{q}} |u|^2_{1,\Omega},$$

since $|\frac{Ku}{|u|_{1,\Omega}}|_{1,\Omega} = K$.

Therefore, whenever $q \geqslant 1$ the following estimate is valid:

$$\int_\Gamma \left|h_K(u)\right|^{\frac{q+1}{q}} dx \leqslant C_h + C_q L_q^{\frac{q+1}{q}}(K)|u|^2_{1,\Omega}. \tag{3.13}$$

We shall analyze in the sequel the case of $q < 1$. From (3.10), considering $R = K$, we obtain

$$\int_\Gamma \left|h_K(u)\right|^{\frac{q+1}{q}} dx \leqslant C_q L_q(K)^{\frac{q+1}{q}} |u|^{\frac{q+1}{q}}_{1,\Omega} + C_h, \tag{3.14}$$

whenever $|u|_{1,\Omega} \leqslant K$. If, instead, $q < 1$ and $|u|_{1,\Omega} \geqslant K$, the definition of h_K and the fact that $K \geqslant 1$ imply

$$\begin{aligned}\int_\Gamma \left|h_K(u)\right|^{\frac{q+1}{q}} dx &\leqslant L_q(K)^{\frac{q+1}{q}} \left(\frac{|Ku|_{1,\Omega}}{|u|_{1,\Omega}}\right)^{\frac{q+1}{q}} + C_h \\ &\leqslant C_q L_q(K)^{\frac{q+1}{q}} K^{\frac{1-q}{q}} |u|^2_{1,\Omega} + C_h \leqslant C_q L_q(K)^{\frac{q+1}{q}} K^{\frac{q+1}{q}} |u|^2_{1,\Omega} + C_h.\end{aligned}$$

In any case we have

$$\int_\Gamma \left|h_K(u)\right|^{\frac{q+1}{q}}\,dx \leqslant C_q L_q(K)^{\frac{q+1}{q}} K^{\frac{q+1}{q}} |u|_{1,\Omega}^2 + C_q L_q(K)^{\frac{q+1}{q}} |u|_{1,\Omega}^2 + C_h. \tag{3.15}$$

Selecting $\epsilon = m_q$, and recalling (3.8), we obtain from (3.9) along with (3.15)

$$\begin{aligned}\left|U(t)\right|_H^2 \leqslant \left|U(0)\right|_H^2 + L_f^2(K)\int_0^t \left|U(s)\right|_H^2\,ds + C_{g,h}t \\ + C_{\epsilon,q}\big(L_q^{\frac{q+1}{q}}(K)\big(1 + K^{\frac{q+1}{q}}\big)\big)\int_0^t \left|u(s)\right|_{1,\Omega}^2\,ds.\end{aligned}$$

Gronwall's inequality gives for all $t < T$

$$\left|u(t)\right|_{1,\Omega}^2 + \left|u_t(t)\right|_{0,\Omega}^2 \leqslant \big[\left|u(0)\right|_{1,\Omega}^2 + \left|u_t(0)\right|_{0,\Omega}^2 + C_{g,h}\,T\big]e^{C(q,m_q)(L_f^2(K)+L_q^{\frac{q+1}{q}}(K)(1+K^{\frac{q+1}{q}}))t}. \tag{3.16}$$

As before, taking K large enough so that $|U(0)|_H^2 + C_{g,h}T < K$, there exists $T_K = T_K(m_q, L_q(K), K, L_f(K))$ such that for all $t < T_K$, $|u(t)|_{1,\Omega}^2 \leqslant K$.

Thus, for $t < T_K$, h_K and f_K coincide with h and f. Consequently, the solution of K problem (9.34) coincides on T_K with the solution of the original equation. Repeating the procedure with a larger K we obtain maximality property existence of the maximal survival time T_M that depends on m_q, $|U(0)|$, $L_f(|U(0)|)$, $L_q(|U(0)|)$. □

4. Proof of Theorem 2.2

We note that the result of Theorem 2.1 provides local solutions $U(t)$ with the length of maximal time T_M depending on the constant m_1 (or more generally m_q) and not on the strong monotonicity constant m_0. This is critical, since, otherwise, the length of maximal time of survival of solution will depend on strong monotonicity. Instead of the dependence on strong monotonicity, the maximal survival time T_M depends only on m_1, that is, depends on condition (3.4). For this reason it becomes clear that assuming growth conditions on $g(s)s$ only at *infinity*, should suffice to obtain weak solutions. In that process the uniqueness of solutions may be however lost.

Proceeding with the proof, we aim to relinquish the condition that g is strongly monotone and that $\hat{h}$ is locally Lipschitz from $H^1(\Omega) \to L_2(\Gamma)$, replacing this latter condition by the condition in Assumption 2.1. It is convenient to consider two separate cases: $q \geqslant 1$ and $q < 1$.

4.1. Superlinear case

We shall *begin with a more challenging superlinear case* $q \geqslant 1$. In this case the analysis follows closely the approximation idea introduced in [23]. This concerns in approximating the original quantities g and h by functions that satisfy hypotheses of Theorem 2.1. This is to say that the approximation of g should satisfy coercivity requirement in the first condition of that theorem and function $\hat{h}$ should be Lipschitz continuous with values in $L_2(\Gamma)$ (this last condition is automatically satisfied when $q < 1$). Careful treatment when passing to the limit on approximations will lead to the desired solution.

Step 1. Approximation of g and h

Let n be a parameter of approximation destined to go to infinity. Following [23] we introduce the following approximations

- g_n, $g_n(s) \equiv g(s) + \frac{1}{n}s$, $n \to \infty$.
- $\hat{h}_n$, $\hat{h}_n$ are locally Lipschitz from $H^1(\Omega) \to L_2(\Gamma)$ and the following inequalities are satisfied *uniformly* in n:

$$\left|\hat{h}_n(u)\right|_{L_{\frac{q+1}{q}}(\Gamma)} \leqslant C_h + L_q(R)|u|_{1-\epsilon,\Omega}, \quad \text{for } |u|_{1-\epsilon,\Omega} \leqslant R, \tag{4.1}$$

$$\left|\hat{h}_n(u_n) - \hat{h}(u)\right|_{L_{\frac{q+1}{q}}(\Gamma)} \to 0, \quad \text{for } u_n \to u \text{ weakly in } H^{1-\epsilon}(\Omega). \tag{4.2}$$

We note that functions g_n are monotone, continuous. Moreover, g_n is strongly monotone with the constant $m_0 = 1/n \to 0$. As for the approximation of $\hat{h}_n$, such functions can be always constructed considering the regularity imposed on h (see [23, p. 515]).

Thus, Theorem 2.1 can be now applied with g_n, h_n. The result of this theorem holds true for each n with the survival time $T_M(|U_0|_H, m_q)$ (see Corollary 3.2), hence T_M is *uniform* in n. Moreover, from (3.16) we also have a priori bounds for $|U(t)|_H \leqslant K$ where K depends on $|U(0)|_H$ for the time of existence of solutions $T \leqslant T_M$. In what follows we shall need additional bounds for the solution. For this we shall exploit further the m_q growth conditions imposed on $g(s)$. Indeed, going back to energy inequality and exploiting the properties of the approximations along with the growth condition on $g(s)$, $g(s)s \geqslant m_q|s|^{q+1}$, $|s| \geqslant 1$, we obtain

$$\begin{aligned}
&\left|u_n(t)\right|_{1,\Omega}^2 + \left|u_{nt}(t)\right|_{0,\Omega}^2 + m_q \int_0^t \int_\Gamma |u_{nt}|^{q+1}\,dx\,ds + \frac{1}{n}\int_0^t\int_\Gamma |u_{nt}|^2\,dx\,ds - C_g t \\
&\quad \leqslant \left|u(0)\right|_{1,\Omega}^2 + \left|u_{nt}(0)\right|_{0,\Omega}^2 + \int_0^t\int_\Gamma \hat{h}_n(u_n)u_{nt}\,dx\,ds + L_f\big(|U_0|_H\big)\int_0^t \left|u_n(s)\right|_{1,\Omega}\left|u_{nt}(s)\right|_{0,\Omega}ds.
\end{aligned}$$

By Young's inequality and the approximation assumption imposed on $\hat{h}_n$ we infer for $t < T_M$ and any $\epsilon > 0$ that

$$\begin{aligned}
&\left|u_n(t)\right|_{1,\Omega}^2 + \left|u_{nt}(t)\right|_{0,\Omega}^2 + m_q \int_0^t \int_\Gamma |u_{nt}|^{q+1}\,dx\,ds - C_g t \\
&\quad \leqslant \left|u(0)\right|_{1,\Omega}^2 + \left|u_t(0)\right|_{0,\Omega}^2 + \epsilon \int_0^t \int_\Gamma |u_{nt}|^{q+1}\,dx\,ds
\end{aligned}$$

$$+ L_f^2\big(|U_0|_H\big) + L_q^{\frac{q+1}{q}}\big(|U_0|_H\big)(C_\epsilon + 1)\left[\int_0^t \big|u_n(s)\big|_{1,\Omega}^2\, ds + \int_0^t \big|u_{nt}(s)\big|_{0,\Omega}^2\, ds + C_h\right].$$

Choosing $0 < \epsilon < m_q$, recalling Assumption 2.1, in particular bounds in (2.1), and applying Gronwall's inequality give the a priori bounds on $[0, T_M)$,

$$\big|u_n(t)\big|_{1,\Omega} + \big|u_{nt}(t)\big|_{0,\Omega} \leqslant C\big(|U_0|_H, m_q, L_f\big(\big|U(0)\big|_H\big), L_q\big(\big|U(0)\big|_H\big)\big),$$

$$\int_0^t \int_\Gamma \big[\big|g(u_{nt})\big|^{\frac{q+1}{q}} + |u_{nt}|^{q+1}\big]\, dx\, ds \leqslant C\big(m_q, M_q, |U_0|_H, L_q\big(\big|U(0)\big|_H\big), L_f\big(\big|U(0)\big|_H\big)\big). \tag{4.3}$$

Since $q \geqslant 1$ the above estimate trivially implies

$$\int_0^t \int_\Gamma |u_{nt}|^2\, dx\, ds \leqslant C\big(m_q, M_q, |U_0|_H\big), \quad t \in (0, T_M). \tag{4.4}$$

Thus, we have

$$U_n \to U \quad \text{weakly star in } L_\infty\big((0, T_M); H\big),$$
$$\big(g(u_{nt}), u_{nt}\big) \to \big(g^*, u_t\big)$$

weakly as monotone graphs in $L_{q+1}(\Sigma_{T_M}) \times L_{(q+1)/q}(\Sigma_{T_M})$, for $q > 0$, where $\Sigma_{T_M} \equiv \Gamma \times (0, T_M)$. Moreover, from (4.2) we also have that

$$\frac{1}{n}\int_0^{T_M} \int_\Gamma |u_{nt}|^2\, dx\, ds \to 0. \tag{4.5}$$

Step 2. Strong limits

Our main goal is to show that $U(t)$ satisfies variational formulation of the equation. The passage on the limit in the variational form of the equation is straightforward except for the nonlinear terms. Indeed, weak convergence does not suffice to pass with the limit on f, h and g. Moreover, the limit g^* needs to be identified with a correct quantity which should be $g(u_t)$. These are main technical points we will be dealing with below.

The main points are in proving the following strong convergence

$$U_n \to U \quad \text{strongly in } L_\infty\big((0, T_M); H\big) \tag{4.6}$$

along with the estimate

$$\lim_{n,m\to\infty} \int_0^t \int_\Gamma \big(g(u_{nt}) - g(u_{mt})\big)(u_{nt} - u_{mt})\, dx\, dt \to 0, \quad t < T_M. \tag{4.7}$$

Denote $\tilde{U} \equiv U_n - U_m$, where U_n, U_m are strong solutions corresponding to regularized problem (1.1) with $g_n, \hat{h}_n$ and $g_m, \hat{h}_m$ respectively and the same initial data. We note, that both solutions are defined on the same time interval *independent on* m, n. This follows from Corollary 3.2. We apply energy inequality to these strong solutions defined on $[0, T_M]$.

$$\begin{aligned}
&\frac{1}{2}\left|\tilde{U}(t)\right|_H^2 + \int_0^t \int_\Gamma \big(g(u_{nt}) - g(u_{mt})\big)(u_{nt} - u_{mt})\,dx\,ds \\
&\quad \leqslant \frac{3}{2}(1/n + 1/m)\int_0^t \big[|u_{nt}|_{0,\Gamma}^2 + |u_{mt}|_{0,\Gamma}^2\big]\,ds \\
&\qquad + \left(\int_0^t \int_\Gamma |u_{nt} - u_{mt}|^{q+1}\,dx\,ds\right)^{\frac{1}{q+1}} \left(\int_0^t \int_\Gamma \left|\hat{h}_n(u_n) - \hat{h}_m(u_m)\right|^{\frac{q+1}{q}}\,dx\,ds\right)^{\frac{q}{q+1}} \\
&\qquad + \left(\int_0^t \int_\Omega |u_{nt} - u_{mt}|^2\,dx\,ds\right)^{\frac{1}{2}} \left(\int_0^t \int_\Omega \left|\hat{f}(u_n) - \hat{f}(u_m)\right|^2\,dx\,ds\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.8}$$

By (4.3) we have

$$\int_0^t \int_\Gamma |u_{nt}|^{q+1}\,dx\,ds \leqslant C, \quad t < T_M, \tag{4.9}$$

uniformly in n, where $C = C(m_q, M_q, |U_0|_H, L_q, L_f)$.

Hence, from (4.8) and local Lipschitz property: $H^1(\Omega) \to L_2(\Omega)$, of $f(u)$ we infer

$$\begin{aligned}
&\frac{1}{2}\left|\tilde{U}(t)\right|_H^2 + \int_0^t \int_\Gamma \big(g(u_{nt}) - g(u_{mt})\big)(u_{nt} - u_{mt})\,dx\,ds \\
&\quad \leqslant \frac{3}{2}(1/n + 1/m)\int_0^t \big[|u_{nt}|_{0,\Gamma}^2 + |u_{mt}|_{0,\Gamma}^2\big]\,ds \\
&\qquad + C\left(\int_0^t \int_\Gamma \left|\hat{h}_n(u_n) - \hat{h}_m(u_m)\right|^{\frac{q+1}{q}}\,dx\,ds\right)^{\frac{q}{q+1}} + C\int_0^t \left|\tilde{U}(s)\right|_H^2\,ds.
\end{aligned} \tag{4.10}$$

From (4.2) we obtain:

$$\int_0^t \int_\Gamma \left|\hat{h}_n(u_n) - \hat{h}_m(u_m)\right|^{\frac{q+1}{q}}\,dx\,ds$$

$$\leqslant \int_0^t \int_\Gamma \big[\big|\hat{h}_n(u_n) - \hat{h}(u)\big| + \big|\hat{h}_m(u_m) - \hat{h}(u)\big|\big]^{\frac{q+1}{q}}\,dx\,ds. \tag{4.11}$$

Since $U_n \to U$ weakly star in $L^\infty(0, T_M; H)$ we obtain that $u_n \to u$ weakly star in $L^\infty(0, T_M; H^1(\Omega))$.

Then, for $t < T_M$ (4.11) implies

$$\int_0^t \int_\Gamma \big|\hat{h}_n(u_n) - \hat{h}_m(u_m)\big|^{\frac{q+1}{q}}\,dx\,ds \to 0.$$

In addition, recalling (4.5), we get

$$(1/n + 1/m)\int_0^t \big[|u_{nt}|^2_{0,\Gamma} + |u_{mt}|^2_{0,\Gamma}\big]\,ds \to 0.$$

Returning to (4.10) and applying Cronwall's inequality we deduce that

$$\frac{1}{2}\big|\tilde{U}(t)\big|^2_H + \int_0^t \int_\Gamma \big(g(u_{nt}) - g(u_{mt})\big)(u_{nt} - u_{mt})\,dx\,dt \to 0, \tag{4.12}$$

which yields now convergence in (4.4) and (4.7). Convergence in (4.7) along with monotonicity of g and weak limit $(g(u_{nt}), u_{nt}) \to (g^*, u_t)$ allows to conclude by Lemma 1.3 [2] that the limit is strong and $g^* = g(u_t) \in L_{(q+1)/q}(\Gamma \times (0, T_M))$.

To conclude the proof (for the case $q \geqslant 1$) we need to pass to the limit on equation. This step is now straightforward in view of strong convergence in (4.4), continuity of f on $H_1(\Omega)$, convergence of $\hat{h}_n u_n$ to $\hat{h}u$ in $L_{(q+1)/q}(\Gamma)$, and finally the convergence $g(u_{nt}) \to g(u_t)$ in $L_{(q+1)/q}(\Gamma \times (0, T_M))$. The proof of local existence of solutions in the case $q \geqslant 1$ is thus completed.

4.2. Sublinear damping

When $q < 1$ all arguments above are valid with the exception of (4.4) and (4.5). This prevents passing with the limit on strictly monotone regularization of Eq. (1.1). In view of this, when $q < 1$ we neither approximate g nor h. (In fact, when $q < 1$ any function in $L_{(q+1)/q}(\Gamma)$ is automatically in $L_2(\Gamma)$, so there is no need for approximating h.) Instead, our approach in this case is based on a classical Faedo–Galerkin approximation. A convenient finite dimensional space to work with is $V_n = \operatorname{span}\{\phi_1, \ldots, \phi_n\}$ where ϕ_j are eigenfunctions of Laplacian with Robin boundary conditions. We consider the same truncated functions f_K, h_K as in the previous case. We define approximate solutions by the usual variational formula: Find $U_{n,K} \in C([0, T], V_n \times V_n)$ such that $U_{n,K} = (u_n, u_{nt})$, where $u_n(t) = \sum_{i=1}^n \beta_{in}(t)\phi_i$ verifies

$$(u_{ntt}, v)_\Omega + (\nabla u_n, \nabla v)_\Omega + \langle u, v\rangle_\Gamma + \big\langle g(u_{nt}), v\big\rangle_\Gamma = \big(f_K(u_n), v\big)_\Omega + \big\langle h_K(u_n), v\big\rangle_\Gamma \tag{4.13}$$

for all $v \in V_n$. The initial conditions are given as the standard projections of $U(0)$ on $V_n \times V_n$. The existence and uniqueness of solutions to approximating problem defined on some interval $[0, T_{n,K})$ follow now from the classical ODE argument. The key point is that by the estimates of Corollary 3.2, the time $T_{n,K}$ can be made independent on n—say T_K. By selecting T_K appropriately small (according to the same estimate as in the previous case), the solutions with truncated f_K, h_K are the same as with untruncated f, h. The critical part is stability estimates which are obtained as in Step 2 by the arguments almost identical to these in Corollary 3.2. These imply weak convergence as in Step 1 above. The passage to strong limits follows from a subset of arguments in Step 2 above.

Indeed, selecting in (4.13), $v \equiv u_{nt} = \sum_{i=1}^n \beta'_{in}(t)\phi_i$ and applying the same energy type of argument as that leading to (4.3), we obtain the two bounds in (4.3). This implies weak convergence

$$\begin{aligned} &U_n \to U \quad \text{weakly star in } L_\infty(0, T_M; H), \\ &\big(g(u_{nt}), u_{nt}\big) \to \big(g^*, u_t\big) \quad \text{weakly in } L_{q+1}(\Sigma_{T_M}) \times L_{\frac{q+1}{q}}(\Sigma_{T_M}). \end{aligned} \tag{4.14}$$

In order to pass with the limit on the variational form (4.13) we shall follow arguments of Step 2 in the proof of Theorem 2.2 and establish the strong convergence

$$U_n \to U \quad \text{strongly in } L_\infty\big((0, T_M); H\big) \tag{4.15}$$

along with the estimate

$$\lim_{n,m\to\infty} \int_0^t \int_\Gamma \big(g(u_{nt}) - g(u_{mt})\big)(u_{nt} - u_{mt})\, dx\, ds \to 0, \quad t < T_M. \tag{4.16}$$

Indeed, a counterpart of inequality in (4.8) is now

$$\begin{aligned} &\frac{1}{2}\big|\tilde{U}(t)\big|_H^2 + \int_0^t \int_\Gamma \big(g(u_{nt}) - g(u_{mt})\big)(u_{nt} - u_{mt})\, dx\, ds \\ &\quad \leqslant \left(\int_0^t \int_\Omega \big|f(u_n) - f(u_m)\big|^2 dx\, ds\right)^{\frac{1}{2}} \left(\int_0^t \int_\Omega |u_{nt} - u_{mt}|^2\, dx\, ds\right)^{\frac{1}{2}} \\ &\qquad + \left(\int_0^t \int_\Gamma |u_{nt} - u_{mt}|^{q+1}\, dx\, ds\right)^{\frac{1}{q+1}} \left(\int_0^t \int_\Gamma \big|\hat{h}(u_n) - \hat{h}(u_m)\big|^{\frac{q+1}{q}}\, dx\, ds\right)^{\frac{q}{q+1}}. \end{aligned} \tag{4.17}$$

Since by (4.3), $|u_{nt}|_{L_{q+1}(\Sigma_{T_M})}$ is uniformly bounded by the initial data and we also have that

$$\int_0^t \int_\Omega \big|f(u_n) - f(u_m)\big|^2 dx\, ds, \qquad \int_0^t \int_\Gamma \big|\hat{h}(u_n) - \hat{h}(u_m)\big|^{\frac{q+1}{q}}\, dx\, ds \to 0, \quad t < T_M,$$

we obtain the same as in (4.12). This leads to the desired conclusion in (4.15) and (4.16). The convergence in the second statement of (4.14) along with Lemma 1.3 [2] and (4.16) imply

$$g(u_{nt}) \to g(u_t) \quad \text{weakly in } L_{\frac{q+1}{q}}(\Sigma_{T_M}).$$

The above convergence allows the passage to the limit on the nonlinear term $g(u_{nt})$. The passage through the limit on other terms follows from the strong convergence in (4.15). This way we obtain the desired conclusion that the limit function of Galerkin approximations satisfies variational form of the equation, as desired. Thus, the proof of *local existence* of solution announced in the first statement of the theorem is complete.

The second statement on *global existence* of solutions follows automatically from the hypotheses made. Indeed, since the "survival time" of solutions depends on $|U(T_M)|_H$ and the construction of weak solutions does not require additional regularity of solutions, an existence of a priori bound leads to global existence of solutions.

Finally, the last assertion in the theorem: validity of energy identity for *weak* solutions, is a direct consequence of monotonicity of g and assumptions imposed on f, h. Indeed, while the inequality $E(t) \leqslant E(0)$ can be always proved in a nonlinear context due to weak lower semicontinuity of the energy, the actual identity is a more subtle issue. One needs to pass with the limit on the nonlinear dissipative term. However, due to monotonicity of $g(s)$, Lemma 3.1 in [2] (used also in [23]) and relations (4.7), (4.16) one can pass with the limit for strong regular solutions (resulting from monotone operator theory) on the term $\int_0^t \langle g(u_{nt}), u_{nt}\rangle_\Gamma \, dt \to \int_0^t \langle g(u_t), u_t\rangle_\Gamma dt$ where the duality understood with respect to $L_{q+1} \times L_{(q+1)/q}$ topology. All these ingredients have been shown during the course of the proof.

5. Proof of Theorem 2.3

In order to prove the theorem it suffices to prove the a priori bound. This is given by the following lemma.

Lemma 5.1. *Let $U(t)$ be any local solution given by Theorem* 2.2. *Let h and f satisfy growth conditions of Theorem* 2.3. *Then the following a priori bound holds for all $t \leqslant T$, where T is arbitrary*:

$$\big|U(t)\big|_H^2 + \int_\Gamma \big|u(x,t)\big|^{r+1}\,dx \leqslant C\big(\big|U(0)\big|_H\big),$$

C is a bounded function of its arguments and $r \leqslant q$ or $r \leqslant \frac{2q}{q+1}$.

Proof. We add to both sides of Eq. (1.1) potential term $|u|^{r-1}uu_t$. This term is well defined for finite energy solutions (due to Sobolev's embeddings and trace theorem). Standard energy method gives the inequality

$$\frac{1}{2}\big|U(t)\big|_H^2 + \frac{1}{r+1}\int_\Gamma \big|u(t,x)\big|^{r+1}\,dx + m_q\int_0^t\int_\Gamma |u_t|^{q+1}\,dx\,ds - C_g t - m_q\,\mathrm{meas}(\Gamma)T$$

$$\leqslant \frac{1}{2}\left|U(0)\right|_H^2 + \frac{1}{r+1}\int_\Gamma \left|u(0,x)\right|^{r+1} dx + \int_0^t \big(f(u), u_t\big)_\Omega \, ds$$
$$+ \int_0^t \int_\Gamma \big[h(u) + |u|^r\big] u_t \, dx \, ds. \tag{5.1}$$

The boundary terms are estimated by Holder's and Young's inequalities with the sue of growth condition imposed on h:

$$\int_0^t \int_\Gamma \big[h(u) + |u|^r\big] u_t \, dx \, ds \leqslant \epsilon \int_0^t |u_t|_{0,\Gamma}^{q+1} ds + C_{\epsilon,h} \int_0^t \int_\Gamma |u|^{r\frac{q+1}{q}} dx \, ds + C^*_{\epsilon,h} T.$$

Taking ϵ sufficiently small and exploiting growth condition imposed on f we obtain for $N > 0$

$$\frac{1}{2}\left|U(t)\right|_H^2 + \frac{1}{r+1}\int_\Gamma \left|u(t,x)\right|^{r+1} dx + N\int_0^t \int_\Gamma |u_t|^{q+1} dx \, ds$$
$$\leqslant \frac{1}{2}\left|U(0)\right|_H^2 + \frac{1}{r+1}\int_\Gamma \left|u(0,x)\right|^{r+1} dx + L\int_0^t |u|_{1,\Omega} |u_t|_{0,\Omega} \, ds$$
$$+ C_h \int_0^t \int_\Gamma |u|^{r\frac{q+1}{q}} dx \, ds + C_{h,f,g} T. \tag{5.2}$$

For $r \leqslant q$ inequality (5.2) followed by Gronwall's inequality provides the desired estimate in Lemma 5.1. $\square$

When $r \leqslant 2\frac{q}{q+1}$, then we simply use trace theorem to deduce

$$\int_0^t \int_\Gamma |u|^{r\frac{q+1}{q}} dx \, ds \leqslant C_h \int_0^t \int_\Gamma |u|^2 dx + C_h T \leqslant \int_0^t \left|u(s)\right|_{1,\Omega}^2 ds + C_h T,$$

and Gronwall's inequality yields the desired result.

6. Potential well solutions—Proof of Theorem 2.4

We consider functions h and f of a special polynomial structure, that is, $f(s) = |s|^{p-1}s$, $h(s) = |s|^{k-1}s$ where p, k are such that $H^1(\Omega) \subset L_{p+1}(\Omega)$, $H^{1/2}(\Gamma) \subset L_{k+1}(\Gamma)$; $p, k \geqslant 1$. The associated potential energy takes the form

$$E_p(u) \equiv \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right] dx + \int_\Gamma \left[\frac{1}{2}|u|^2 - \frac{1}{k+1}|u|^{k+1}\right] dx$$

and kinetic energy is given by $E_k(v) \equiv \frac{1}{2}\int_\Omega |v|^2\,dx$. The corresponding energy identity with $E(u,u_t) = E_k(u_t) + E_p(u)$ and $E(t) \equiv E(u(t), u_t(t))$ is written as

$$E(t) + \int_0^t \int_\Gamma g(u_t)u_t\,ds\,dt = E(0). \tag{6.1}$$

As it was mentioned before our goal is to determine a set of initial data that is invariant with respect to the flow. To accomplish this we introduce the following notation:

- $B_\Omega \equiv \sup_{u\in H^1(\Omega)} \frac{|u|_{L_{p+1}(\Omega)}}{\sqrt{|\nabla u|^2_{0,\Omega} + |u|^2_{0,\Gamma}}}$, $B_\Gamma \equiv \sup_{u\in H^1(\Omega)} \frac{|u|_{L_{k+1}(\Gamma)}}{\sqrt{|\nabla u|^2_{0,\Omega} + |u|^2_{0,\Gamma}}}$,
- λ_0 is the first positive zero of the function $F'(x)$ where

$$F(x) \equiv \frac{1}{2}x^2 - \frac{1}{p+1}B_\Omega^{p+1}x^{p+1} - \frac{1}{k+1}B_\Gamma^{k+1}x^{k+1},$$

- $d \equiv F(\lambda_0)$.

Let λ_0 be the first positive zero of the derivative $F'(x)$. Then,

$$0 = F'(\lambda_0) = \lambda_0 - B_\Omega^{p+1}\lambda_0^p - B_\Gamma^{k+1}\lambda_0^k, \tag{6.2}$$

and consequently,

$$\lambda_0\big(1 - B_\Omega^{p+1}\lambda_0^{p-1} - B_\Gamma^{k+1}\lambda_0^{k-1}\big) = 0.$$

Since $\lambda_0 > 0$, the last identity yields

$$1 - B_\Omega^{p+1}\lambda_0^{p-1} - B_\Gamma^{k+1}\lambda_0^{k-1} = 0. \tag{6.3}$$

Next, we are going to prove that λ_0 is a point of local maximum. For this end, we will show that $F''(\lambda_0) < 0$. Indeed, since $p > 1$ and $k > 1$ and from (6.3) we deduce that

$$\lambda_0 - pB_\Omega^{p+1}\lambda_0^p - kB_\Gamma^{k+1}\lambda_0^k < \lambda_0 - B_\Omega^{p+1}\lambda_0^p - B_\Gamma^{k+1}\lambda_0^k = 0.$$

The last inequality implies that

$$pB_\Omega^{p+1}\lambda_0^p + kB_\Gamma^{k+1}\lambda_0^k > \lambda_0,$$

and consequently,

$$pB_\Omega^{p+1}\lambda_0^{p-1} + kB_\Gamma^{k+1}\lambda_0^{k-1} > 1. \tag{6.4}$$

From (6.4) we deduce that $F''(\lambda_0) < 0$, as we desired to prove.

Observe that

(i) $F'(x) > 0$ if $0 < x < \lambda_0$,
(ii) $F'(x) < 0$ if $x > \lambda_0$.

In fact, since $0 < x < \lambda_0$, we deduce

$$F'(x) = x\big(1 - B_\Omega^{p+1} x^{p-1} - B_\Gamma^{k+1} x^{k-1}\big) > x\big(1 - B_\Omega^{p+1} \lambda_0^{p-1} - B_\Gamma^{k+1} \lambda_0^{k-1}\big) = 0.$$

The proof of (ii) is analogous.

Since $p > 1$ and $k > 1$, we easily verify that

$$\lim_{x\to+\infty} F(x) = -\infty.$$

Considering the above features, our function has the behaviour given in Fig. 1.

In this context we have the following lemma.

Lemma 6.1. *The set*

$$A \equiv \big\{(u_0, u_1) \in H^1(\Omega) \times L_2(\Omega);\ |\nabla u_0|_{0,\Omega}^2 + |u_0|_{0,\Gamma}^2 \leqslant \lambda_0^2,\ E(0) < d\big\}$$

is invariant under the flow. Moreover, for every $(u_0, 0) \in A$ we have that

$$|\nabla u_0|_{0,\Omega}^2 + |u_0|_{0,\Gamma}^2 \leqslant C_{p,q} E_p(u_0).$$

Proof. Let $(u(t), u_t(t)) = T(t)(u_0, u_1)$ where $T(t)$ denotes nonlinear flow. From (6.1) it follows immediately that

$$E_p\big(u(t)\big) \leqslant E\big(u(t), u_t(t)\big) \leqslant E(u_0, u_1) < d. \tag{6.5}$$

On the other hand, using the definition of B_Ω, B_Γ and the following notation

$$|u|_{H^1} \equiv \sqrt{|\nabla u_0|_{0,\Omega}^2 + |u_0|_{0,\Gamma}^2}$$

we also obtain

$$E_p(u) \geqslant \frac{1}{2}|u|_{H^1}^2 - \frac{1}{p+1} B_\Omega^{p+1} |u|_{H^1}^{p+1} - \frac{1}{k+1} B_\Gamma^{k+1} |u|_{H^1}^{k+1} = F\big(|u|_{H^1}\big). \tag{6.6}$$

Hence, for all solutions originating in A we have

$$F\big(\big|u(t)\big|_{H^1}\big) < d.$$

Since $F(0) = 0$, (λ_0, d) is a point of maximum of the graph $F(x)$, $|u_0|_{H^1} < \lambda_0$, by continuity in time of $F(|u(t)|_{H^1})$ the graph of $F(|u(t)|_{H^1})$ cannot reach (λ_0, d). So we must have $|u(t)|_{H^1} < \lambda_0$. The above relation together with (6.5) gives the first statement in the lemma.

As for the second statement, we simply note that for $\lambda < \lambda_0$ we have

$$F(\lambda)=\lambda^2\left[\frac{1}{2}-\frac{1}{p+1}B_\Omega^{p+1}\lambda^{p-1}-\frac{1}{k+1}B_\Gamma^{k+1}\lambda^{k-1}\right]$$
$$\geqslant\lambda^2\left[\frac{1}{2}-\frac{1}{p+1}B_\Omega^{p+1}\lambda_0^{p-1}-\frac{1}{k+1}B_\Gamma^{k+1}\lambda_0^{k-1}\right].$$

Using equation $F'(\lambda_0)=0$ we obtain

$$\lambda_0=B_\Omega^{p+1}\lambda_0^p+B_\Gamma^{k+1}\lambda_0^k \quad\Longrightarrow\quad 1=B_\Omega^{p+1}\lambda_0^{p-1+B_\Gamma^{k+1}\lambda_0^{k-1}}$$

Thus, for $k\geqslant p$

$$F(\lambda)\geqslant\lambda^2\left[\frac{1}{2}-\frac{1}{p+1}+\left(\frac{1}{p+1}-\frac{1}{k+1}\right)B_\Gamma^{k+1}\lambda_0^{k-1}\right]\geqslant c_0\lambda^2,$$

where $c_0=\frac{1}{2}-\frac{1}{p+1}$. If $k\leqslant p$ we have the same inequality with $c_0=\frac{1}{2}-\frac{1}{k+1}$. Since $|u_0|_{H^1}<\lambda_0$ we obtain

$$E_p(u_0)\geqslant F\big(|u_0|_{H^1}\big)\geqslant c_0|u_0|_{H^1}^2,$$

the conclusion of the lemma follows with $C_{p,k}=\frac{1}{c_0}$. □

Corollary 6.2. *Under the assumption of Theorem* 2.2*, solutions corresponding to the initial data* $(u_0,u_1)\in A$ *are global. Moreover* $T(t)A\subset A$ *and* $|u(t)|_{1,\Omega}\leqslant CE_p(u(t))\leqslant Cd$.

Inequality in (6.5) and the result of the corollary above imply all the statements of Theorem 2.4.

7. Uniform decay rates—Proof of Theorem 2.6

We work now with a special class of solutions for which one has not only global existence but, in addition, boundedness of solutions for positive times. Thus, under Assumption 2.3 and hypotheses of Theorem 2.6 we have the following well-posedness result:

Lemma 7.1. *Weak solutions referred to in Theorem* 2.6 *have the following properties*:

$$u\in C\big([0,\infty),H^1(\Omega)\big)\cap C^1\big([0,\infty),L_2(\Omega)\big), \tag{7.1}$$
$$u_t,\partial_\nu u,\nabla_\sigma u\in L_2\big(0,\infty;L_2(\Gamma)\big) \tag{7.2}$$

where ∇_σ *denotes tangential derivative.*

Proof. The regularity in the first statement follows from Theorem 2.2, the a priori bound implied by the energy identity in (2.6) and the assumption that solutions considered satisfy $|u(t)|_{1,\Omega}^2\leqslant CE(t)$. As for the boundary regularity in (7.2), this follows from: (i) (7.4) which implies that $\int_0^\infty|u_t(t)|_{0,\Gamma}^2\leqslant C_gE(0)$, (ii) $h(u)\in L_2(0,\infty;L_2(\Gamma))$, implied by (7.1) and Assumption 2.3, (iii) tangential regularity on the boundary of solutions to wave equation [28]. In fact, it is shown in [28] that functions subject to (7.1) and such that $u_{tt}-\Delta u\in L_1(0,T;L_2(\Omega))$,

$\partial_\nu u, u_t \in L_2(0,\infty; L_2(\Gamma))$ possess additional tangential regularity $\nabla_\sigma u \in L_2(0,\infty; L_2(\Gamma))$. Since $f(u) \in C(0,T; L_2(\Omega))$ and we already know that $\partial_\nu u, u_t \in L_2(0,\infty; L_2(\Gamma))$, the premise of the previous statement does hold true. So, the tangential derivative has the regularity claimed. $\square$

One of the issues in proving decay rates of solutions to nonlinear PDEs is the lack of sufficient regularity of solutions that is needed to justify differential calculus. Indeed, the regularity of "weak" solutions is often insufficient to run the computations. In the case when one has uniqueness of weak solutions and sufficient regularity of strong solutions, typical procedure is to perform differential calculus on strong solutions and the final decay estimates are recovered via the limit process. In our case this route is non-applicable due to intrinsic non-uniqueness of solutions. In addition, strong solutions may not have sufficient regularity (note that function g is not assumed differentiable). In view of this a different route around is needed. In fact, we shall use the same procedure as in [23] where solutions are approximated by a sequence of functions with prerequisite regularity (not necessarily solutions to the nonlinear problem). The key starting point to this construction is boundary regularity of the original solutions as announced in Lemma 7.1. In what follows below we shall recall regularization procedure from [23] given in Lemma 2.2.

Lemma 7.2. *(See [23,28].) Let u be any given function with regularity as in Lemma* 7.1. *Then, there exists a sequence of functions*

$$u_l \in C\big([0,T]; H^2(\Omega)\big) \cap C^1\big([0,T]; H^1(\Omega)\big), \qquad u_{ltt} - \Delta u_l \in C\big([0,T]; H^1(\Omega)\big)$$

such that

$$\begin{aligned} &u_l \to u \quad \text{in } C\big([0,T]; H^1(\Omega)\big) \cap C^1\big([0,T]; L_2(\Omega)\big),\\ &(\partial_\nu u_l, u_{tl}, \nabla_\sigma u_l) \to (\partial_\nu u, u_t, \nabla_\sigma u) \quad \text{in } \big[L_2\big(0,T; L_2(\Gamma)\big)\big]^3,\\ &f_l \equiv u_{ltt} - \Delta u_l \to f \equiv u_{tt} - \Delta u \quad \text{in } L_1\big(0,T; L_2(\Omega)\big). \end{aligned} \tag{7.3}$$

The idea of approximation–regularization is that any weak solutions with the properties as in Lemma 7.1 can be approximated by functions u_l which are smooth and which satisfy the obvious equation:

$$u_{ltt} - \Delta u_l = f_l.$$

This is a linear equation with a forcing term f_l for which we will be running computations. Only at the final stage of inequalities in PDE lemmas we shall pass with the limit reconstructing the original solution u. In this procedure we have approximation for each (potentially non-unique) solution. This is of course a different point of view from the classical one when one approximates the initial data of solutions rather than the solution itself. When approximating initial data only, one cannot assert that the limit is a given non-unique solution we want to have a claim for. In fact, a first application of approximation Lemma 7.2 is the validity of energy identity for all weak solutions subject to regularity requirements in Lemma 7.1.

Remark 7.1. We note that while energy identity has been already obtained for weak solutions given in Theorem 2.2, due to the non-uniqueness it is not clear that every weak solution will satisfy the energy relation. The goal of the next corollary is to assert that every weak solution with the boundary regularity as posted does indeed satisfy the energy relation.

Corollary 7.3. *Let u be any weak solution to* (1.1) *such that regularity posted in Lemma* 7.1 *holds. Then we have the energy identity*:

$$E(t) + \int_s^t \int_\Gamma g(u_t) u_t \, dx \, dz = E(s), \quad 0 \leqslant s \ll t < T_M. \tag{7.4}$$

Proof. It follows from Lemma 7.2. Indeed, let u be any weak solution and let u_l be its approximant. We consider $u_{ltt} - \Delta u_l = f_l$ for which we apply standard energy method (multiply by u_{lt} and integrate by parts). This gives

$$\begin{aligned} &|u_{lt}(t)|^2_{0,\Omega} + |\nabla u_l(t)|^2_{0,\Omega} - 2\int_0^t \int_\Gamma \partial_\nu u_l u_{lt} \, dx \, ds \\ &\quad = |u_{lt}(0)|^2_{0,\Omega} + |\nabla u_l(0)|^2_{0,\Omega} + 2\int_0^t \int_\Omega f_l u_{lt} \, dx \, ds. \end{aligned}$$

On the strength of Lemma 7.2 we pass with a strong limit obtaining

$$|u_t(t)|^2_{0,\Omega} + |\nabla u(t)|^2_{0,\Omega} - 2\int_0^t \int_\Gamma \partial_\nu u u_t \, dx \, ds = 2\int_0^t \int_\Omega f u_t \, dx \, ds + |u_t(0)|^2_{0,\Omega} + |\nabla u(0)|^2_{0,\Omega}.$$

Noting that for weak solutions we have $f = f(u) \in C([0, T]; L_2(\Omega))$ and $\partial_\nu u = -u + h(u) - g(u_t) \in L_2(\Sigma)$ we obtain

$$\begin{aligned} &|u_t(t)|^2_{0,\Omega} + |\nabla u(t)|^2_{0,\Omega} + 2\int_0^t \int_\Gamma \big(u - h(u) + g(u_t)\big) u_t \, dx \, ds \\ &\quad = 2\int_0^t \int_\Omega f u_t \, dx \, ds + |u_t(0)|^2_{0,\Omega} + |\nabla u(0)|^2_{0,\Omega}. \end{aligned}$$

The above relation implies the conclusion desired. $\square$

We begin with few elementary relations linking the nonlinear energy function with a classical "linear" energy denoted by $E_0(u,v) = 1/2[|\nabla u|^2_{0,\Omega} + |u|^2_{0,\Gamma}] + E_k(v)$.

$$\begin{cases} E_0(u(t),v(t)) \leqslant cE(u(t),v(t)), \\ E(u(t),v(t)) \leqslant E_0(u(t),v(t)) + C_\delta(E_0(u(t),v(t)))|u(t)|^2_{1-\delta,\Omega}, \\ E(u(t),v(t)) \leqslant C(E_0(u(t),v(t))), \end{cases} \tag{7.5}$$

where $C(s)$ denotes an increasing function that is bounded for bounded arguments and $\delta > 0$ is a suitably small constant. The first inequality follows from (2.9) while the second inequality follows from Sobolev's embeddings and the fact that $\mathcal{F}(s) \leqslant C[s^2 + |s|^{k_0+1}]$ and $\mathcal{H}(s) \leqslant C[|s|^2 + |s|^{k_1+1}]$ are, respectively, locally Lipschitz from $H^{1-\delta}(\Omega) \to L_1(\Omega)$ and $H^{1/2-\delta}(\Gamma) \to L_1(\Gamma)$ for a suitably small δ. The third inequality follows immediately from the definition of $E(u,v)$ and Sobolev's embeddings.

The following lemma is critical:

Lemma 7.4. *Let u be any weak solution to problem* (1.1) *subject to the regularity given in Lemma* 7.1. *We also assume Assumption* 2.3. *Let $\alpha > 0$ be any constant such that $\alpha < \frac{1}{2}T$. Then, there exists $\delta > 0$, $\epsilon > 0$ and constants $C, C_{\alpha,T}(E(0)), C_{\alpha,T,\epsilon,\delta}(E(0))$ such that*

$$\int_\alpha^{T-\alpha} E_0(t)\,dt \leqslant CE(0) + C_{\alpha,T}\big(E(0)\big)\int_0^T\int_\Gamma \left[|u_t|^2 + \left|\frac{\partial u}{\partial \nu}\right|^2\right]dx\,dt$$
$$+ C_{\alpha,T,\delta,\epsilon}\big(E(0)\big)\int_0^T |u|^2_{1-\delta,\Omega}\,dt + \epsilon\int_0^T |u_t|^2_{0,\Omega}\,dt$$

where the first constant C is independent on T.

Proof. *Step 1.* We begin with a standard multiplier inequality obtained first for smooth approximations of weak solutions, functions u_l from Lemma 7.2. Passage with the limit on these approximations gives:

$$\int_0^T \big[\big|u_t(t)\big|^2_{0,\Omega} + \big|\nabla u(t)\big|^2_{0,\Omega}\big]\,dt \leqslant C\big[\|u_t\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u\|^2_{L^\infty(0,T;H^1(\Omega))}\big]$$
$$+ C\int_\Sigma \big[|u|^2 + |u_t|^2 + |\partial_\nu u|^2 + |\nabla_\sigma u|^2\big]\,d\Sigma + C\left|\int_Q \big(f(u)m(x)\nabla u + f(u)u\big)\,dQ\right|, \tag{7.6}$$

where the constant C is independent on T and ∇_σ stands for a tangential derivative. As mentioned, formal derivation of above inequality is by now very standard and obtained with the classical multipliers $2m\cdot\nabla u$ and $(n-1)u$ where $m = (x - x_0)$. These calculations require regularity of solutions higher than available for weak solutions. For this reason we perform calculations on approximants u_l given in Lemma 7.2. The final step is passage with the limit. On the strength of Lemma 7.1 and approximation Lemma 7.2 we pass with the strong limit to obtain (7.6).

Step 2. By inspecting the right-hand side of (7.6) we can immediately identify potentially troublesome spots. These are: (i) tangential derivatives on the boundary and (ii) the last nonlinear term in the inequality that is not compact. (Note we allow critical exponents for f.) Both terms cannot be bounded by terms below the energy level. As we shall see in a moment, the nonlinear term $\int_Q f(u)m(x)\nabla u\,dQ$ can be bounded by, again, tangential derivatives on the boundary and the lower order terms. Indeed, we have,

$$\int_0^T\int_\Omega f(u)(m\cdot\nabla u)\,dx\,dt=\int_\Sigma (m\cdot\nu)\mathcal{F}(u)\,d\Sigma-\int_Q \mathcal{F}(u)\,\mathrm{div}\,m\,dQ.$$

From Assumption 2.3, the embedding $L_{k_0+1}(\Omega)\supset H^{1-\delta}(\Omega)$, (7.4) and (7.5) we deduce that

$$E_0(t)\leqslant cE(t)\leqslant cE(0)\leqslant C\big(E_0(0)\big), \tag{7.7}$$

and for some small $\delta_1>0$

$$\begin{aligned}\int_Q\big|\mathcal{F}(u)\big|\,dQ&\leqslant C\int_Q\big(|u|^2+|u|^{k_0+1}\big)\,dQ\leqslant C\int_0^T\big(\big|u(t)\big|^2_{0,\Omega}+\big|u(t)\big|^{k_0+1}_{1-\delta,\Omega}\big)\,dt\\&\leqslant C_{\delta_1}\big(E(0)\big)\int_0^T\big|u(t)\big|^2_{1-\delta_1,\Omega}\,dt.\end{aligned}\tag{7.8}$$

Similar estimate holds for the term $\int_0^T\int_\Omega f(u)u\,dx\,dt$.

$$\int_Q\big|f(u)\big||u|\,dQ\leqslant C\int_Q\big(|u|^2+|u|^{k_0+1}\big)\,dQ\leqslant C_{\delta_1}\big(E(0)\big)\int_0^T\big|u(t)\big|^2_{1-\delta_1,\Omega}\,dt. \tag{7.9}$$

On the other hand, from Assumption 2.3, we also have the embedding, which however is not compact,

$$H^{1/2}(\Gamma)\hookrightarrow L^{\frac{2n-2}{n-2}}(\Gamma)\hookrightarrow L^{k_0+1}(\Gamma). \tag{7.10}$$

Making analogous computations as in (7.8) and taking (7.10) into account, we infer

$$\begin{aligned}\int_\Sigma\big|\mathcal{F}(u)\big|\,d\Sigma&\leqslant C\int_\Sigma\big(|u|^2+|u|^{k_0+1}\big)\,d\Sigma\leqslant C\int_0^T\big(\big|u(t)\big|^2_{0,\Gamma}+\big|u(t)\big|^{k_0+1}_{1/2,\Gamma}\big)\,dt\\&\leqslant C\int_0^T\big|u(t)\big|^2_{0,\Gamma}\,dt+C\int_0^T\big|u(t)\big|^{k_0-1}_{1/2,\Gamma}\big|u(t)\big|^2_{1/2,\Gamma}\,dt.\end{aligned}\tag{7.11}$$

Interpolation inequality gives

$$|u|_{1/2,\Gamma}^2 \leqslant C|u|_{1,\Gamma}|u|_{0,\Gamma} \leqslant C\big(|u|_{0,\Gamma}^2 + |\nabla_\sigma u|_{0,\Gamma}^2\big).$$

Hence

$$\int_\Sigma \mathcal{F}(u)\,d\Sigma \leqslant C\int_0^T \big|u(t)\big|_{0,\Gamma}^2\,dt + C\|u\|_{L^\infty(0,T;H^1(\Omega))}^{k_0-1}\int_0^T \big(\big|\nabla_\sigma u(t)\big|_{0,\Gamma}^2 + \big|u(t)\big|_{0,\Gamma}^2\big)\,dt. \tag{7.12}$$

Combining (7.8), (7.9) and (7.12) we obtain

$$\begin{aligned}\int_0^T E_0(t)\,dt \leqslant{}& CE(0) + C\int_0^T \big|u_t(t)\big|_{0,\Gamma}^2\,dt \\ &+ C\int_0^T \big[\big|\partial_\nu u(t)\big|_{0,\Gamma}^2 + C\big(E(0)\big)|\nabla_\sigma u|_{0,\Gamma}^2 + C_{\delta_1}\big(E(0)\big)\big|u(t)\big|_{1-\delta_1,\Omega}^2\big]\,dt \\ &+ C\big(E(0)\big)\int_0^T \big|u(t)\big|_{0,\Gamma}^2\,dt,\end{aligned} \tag{7.13}$$

where C is a positive constant.

We point out that the term $|u(t)|_{0,\Gamma}^2$ can be absorbed by $|u(t)|_{1-\delta_1,\Omega}^2$, where $\delta > 0$ is suitably small, and the resulting inequality is also valid in the interval $(\alpha, T-\alpha)$.

Step 3. Thus, the only terms to dispense with are tangential derivatives. For this we employ Lemma 7.2 in Lasiecka and Triggiani [24] which gives: for any $\alpha > 0$, $\alpha < \frac{T}{2}$, and any $\beta > 0$ there exists a constant $C_{T,\alpha,\beta}$ such that

$$\int_\alpha^{T-\alpha}\int_\Gamma |\nabla_\sigma u|^2\,d\Gamma\,dt \leqslant C_{\alpha,\beta,T}\Bigg[\int_\Sigma\bigg(\bigg|\frac{\partial u}{\partial\nu}\bigg|^2 + |u_t|^2\bigg)\,d\Sigma + \|u\|_{H^{1/2+\beta}(Q)}^2 + \big\|f(u)\big\|_{H^{-1/2+\beta}(Q)}^2\Bigg]. \tag{7.14}$$

Note that for all $\beta > 0$ there exist $r < 2$ and $\delta_2 > 0$ such that

$$\int_0^T \big|f(u)\big|_{-1/2+\beta,\Omega}^2 \leqslant \int_0^T \big|f(u)\big|_{L_r(\Omega)}^2 \leqslant C_{\delta_2}\big(E(0)\big)\int_0^T \big|u(t)\big|_{1-\delta_2,\Omega}^2\,dt.$$

Similarly, with any $\epsilon > 0$,

$$|u|^2_{H^{1/2+\beta}(Q)} \leqslant \epsilon \int_0^T |u_t|^2_{0,\Omega}\, dt + (C_\epsilon + \varepsilon) \int_0^T |u|^2_{0,\Omega}\, dt + \varepsilon \int_0^T |\nabla u|^2_{0,\Omega}\, dt.$$

Observe that

$$\int_0^\alpha |\nabla u|^2_{0,\Omega}\, dt + \int_{T-\alpha}^T |\nabla u|^2_{0,\Omega}\, dt \leqslant 2\alpha C\big(E(0)\big).$$

Applying the above inequalities in (7.14) and (7.13) we conclude the desired result. So, Lemma 7.4 is proved with $\delta = \min(\delta_i)$, $i = 1, 2$, and $\varepsilon > 0$ sufficiently small. □

Step 4. By using (7.5) and dissipativity relation we obtain from Lemma 7.4 the following corollary.

Lemma 7.5. *Under the assumptions of Lemma* 7.4

$$\int_0^T E_0(t)\, dt \leqslant C_T\big(E(0)\big)\Bigg[\int_0^T \big|u(t)\big|^2_{0,\Omega}\, dt + \int_\Gamma \big[\big|g(u_t)\big|^2 + |u_t|^2\big]\, dx\, dt\Bigg].$$

Proof. From (7.5) and dissipativity relation (7.4) we easily obtain

$$TE(T) \leqslant \int_0^T E(t)\, dt \leqslant \int_0^T E_0(t)\, dt + C_\delta\big(E(0)\big) \int_0^T \big|u(t)\big|^2_{1-\delta,\Omega}\, dt.$$

On the other hand, from Lemma 7.4, after using boundary conditions and (7.10) in order to obtain

$$\int_0^\alpha E_0(t)\, dt + \int_{T-\alpha}^T E_0(t)\, dt \leqslant 2C\alpha C\big(E(0)\big),$$

we infer

$$\int_0^T E_0(t)\, dt \leqslant (C + 2C\alpha)E(0) + C_T\big(E(0)\big) \int_0^T \int_\Gamma \big[|u_t|^2 + \big|g(u_t)\big|^2 + \big|h(u)\big|^2\big]\, dx\, dt$$
$$+ C_{T,\delta,\alpha,\epsilon}\big(E(0)\big) \int_0^T |u|^2_{1-\delta,\Omega}\, dt + \epsilon \int_0^T |u_t|^2_{0,\Omega}\, dt$$

$$\leqslant CE(T-\alpha)+C_T\big(E(0)\big)\int_0^T\int_\Gamma\big[|u_t|^2+\big|g(u_t)\big|^2\big]\,dx\,dt$$
$$+C_{T,\alpha,\epsilon,\delta}\big(E(0)\big)\int_0^T|u|^2_{1-\delta,\Omega}\,dt+\epsilon\int_0^T|u_t|^2_{0,\Omega}\,dt,$$

where in the last step we have used subcriticality of h to deduce that there exists $\delta>0$ such that

$$\int_\Gamma\big|h\big(u(t)\big)\big|^2\,dx\leqslant C_\delta\big(E(0)\big)\int_0^T|u|^2_{1-\delta,\Omega}\,dt.$$

Appealing again to (7.9) and (7.10), combining the last three inequalities and considering ε sufficiently small lead to

$$\big(T-(C+2C\alpha)\big)E(T)\leqslant C\big(E(0)\big)\int_0^T\int_\Gamma\big[|u_t|^2+\big|g(u_t)\big|^2\big]\,dx\,dt$$
$$+C_{T,\alpha,\delta,\epsilon}\big(E(0)\big)\int_0^T|u|^2_{1-\delta,\Omega}\,dt\,dt. \tag{7.15}$$

Taking T large enough we conclude

$$\int_0^T E(t)\,dt\leqslant C_T\big(E(0)\big)\int_0^T\int_\Gamma\big[|u_t|^2+\big|g(u_t)\big|^2\big]\,dx\,dt$$
$$+C_{T,\alpha,\delta,\epsilon}\big(E(0)\big)\int_0^T|u|^2_{1-\delta,\Omega}\,dt\,dt. \tag{7.16}$$

Using interpolation inequality on the term $|u|^2_{1-\delta,\Omega}$ leads to the desired conclusion in the lemma.

Step 5. Our final step is to absorb the lower order term.

Lemma 7.6. *Let $T>0$ be large enough. Then*

$$\int_0^T\big|u(t)\big|^2_{0,\Omega}\,dt\leqslant C\big(E(0)\big)\int_0^T\int_\Gamma\big[|u_t|^2+\big|g(u_t)\big|^2\big]\,dx\,dt. \tag{7.17}$$

Proof. The proof is identical as in [23]. Note that the uniqueness for stationary problem is guaranteed by the assumption imposed by Theorem 2.5. □

Combining the inequalities in Lemmas 7.5 and 7.6 we obtain the following result.

Lemma 7.7. *For a suitably large* $T > 0$, *the weak solutions of problem* (1.1) *verifies*

$$E(T) \leqslant C\big(T, E(0)\big) \int\limits_{\Sigma_1} \big(\big|g(u_t)\big|^2 + |u_t|^2\big)\, d\Sigma, \tag{7.18}$$

where $E(t)$ *is the energy associated to problem* (1.1).

The completion of the proof follows along the arguments identical to those in [23]. Let $h^* : \mathbb{R}_+ \to \mathbb{R}$ be a concave, strictly increasing function verifying $h^*(0) = 0$ and, for some $N > 0$,

$$h^*\big(sg(s)\big) \geqslant s^2 + \big(g(s)\big)^2, \quad \text{if } |s| \leqslant N. \tag{7.19}$$

Note that (7.19) makes sense in virtue of hypothesis (1) in Assumption 2.3. Define

$$\tilde{h}(x) = h^*\left(\frac{x}{\operatorname{meas}(\Sigma_1)}\right), \quad x \geqslant 0,$$

where $\Sigma_1 = \Gamma_1 \times (0, T)$ and T is given by Lemma 4.4. Setting

$$K := \frac{1}{C(T, E(0)) \operatorname{meas}(\Sigma_1)} \quad \text{and} \quad C := \frac{M_1 + m_1^{-1}}{\operatorname{meas}(\Sigma_1)} \tag{7.20}$$

where m_1 and M_1 are given in Assumption 2.3, we define

$$p(x) := (CI + \tilde{h})^{-1}(Kx). \tag{7.21}$$

We observe that p is well defined since $\tilde{h}$ is monotone increasing and, consequently, $CI + \tilde{h}$ is invertible. In addition, p is a positive, continuous, strictly increasing function with $p(0) = 0$. In the sequel, let us define

$$q^*(x) := x - (I + p)^{-1}(x). \tag{7.22}$$

Then, $q^*(0) = 0$, q^* is strictly increasing and $q^*(x) > 0$ if $x > 0$.

By Lemma 3.2 in [23] and from Lemma 7.7 we have

Lemma 7.8. *Let* p *be defined by* (7.21) *and consider* $T > 0$ *large. Then, if* $E(t)$ *is the corresponding energy to problem* (1.1), *where solutions are considered, we have*

$$p\big(E(T)\big) + E(T) \leqslant E(0).$$

The above inequality implies, as in [23] the final conclusion

$$E(t) \leqslant S(t), \quad t > T_0.$$

This completes the proof of Theorem 2.1.

8. Effective computations of the decay rates given by Theorem 2.6

The algorithm for computations of decay rates given by Theorem 2.6 is very general and provides explicit decay rates without any restrictions on the growth of the dissipation g at the origin. Indeed, as shown in [23] this algorithm gives exponential decay rates for the damping that is bounded from below by a linear function and algebraic decay rates for polynomially decaying dissipation at the origin. We shall illustrate below how other cases can be treated as well. By specializing a bit further the class of nonlinear dissipation we will be able to obtain explicit description of the decay rates. The obtained decay rates are optimal, since they are the same as these optimal rates derived in [1] for the model that does not account for the sources. In addition, we will be able to obtain decay rates for nondifferentiable dissipation, such as fractional powers.

In order to proceed, we note that the behaviour of the function $q^*(s)$ at the origin (this is the only relevant region for the decay rates) is asymptotically equivalent to $(h^*)^{-1}(s)$, where, as we recall, the concave and monotone increasing function $h^*(s)$ is determined from the relation $s^2 + g^2(s) \leqslant h^*(s(g(s)))$, $s \leqslant s_0 < 1$. The fact that such function always exists follows from the monotonicity of $g(s)$, as shown in [23]. Thus the only issue is to determine the structure of $(h^*)^{-1}$ near the origin. Also, it suffices to restrict the analysis to positive values of s. In line with Theorem 2.6 the equation to consider is $S_t + c_0(h^*)^{-1}(c_1 S) = 0$, $S(0) = E(0)$ and the solution of this equation gives an asymptotic bound for the energy function. This is to say we have $E(t) \leqslant C(E(0))S(t)$, for $t > T_0$. The constants c_0, c_1 account for the fact that $q^*(s) \sim (CI + h^*)^{-1}(s)$ at the origin. Indeed, this asymptotic behavior is a direct consequence of the algorithm (7.21), (7.22),

$$\begin{aligned} q^* &= I - (I+p)^{-1} = p \circ (I+p)^{-1} = p \circ \big[\big(p^{-1} + I\big) \circ p\big]^{-1} \\ &= p \circ \big[\big(K^{-1}(CI + \tilde{h}) + I\big) \circ p\big]^{-1} = K^{-1}(CI + \tilde{h})^{-1}. \end{aligned} \tag{8.1}$$

Since $h^*(s) \geqslant cs$, near the origin, for some positive constant c, (8.1) implies $q^*(s) \sim (CI + h^*)^{-1}(s) \geqslant c_1(h^*)^{-1}$ at the origin. Thus, the asymptotic behavior of the energy is driven by the following ODE: $S_t + c_0(h^*)^{-1}(c_1 S) = 0$, $S(0) = E(0)$, as claimed above.

In order to be more specific we consider two cases: (i) $g(s)$ decays to zero faster than linear function, and (ii) $g(s)$ decays slower than linear function. In the first case it suffices to determine $h^*(s)$ from the inequality $s^2 \leqslant h^*(sg(s))$, while in the second case we must have that $g^2(s) \leqslant h^*(sg(s))$.

Solving explicitly $s^2 = h^*(sg(s))$ we obtain that $(h^*)^{-1}(s) = \sqrt{s}g(\sqrt{s})$. For this function to be "eligible" we must verify its concavity, or equivalently convexity of $(h^*)^{-1}(s) = \sqrt{s}g(\sqrt{s})$ that needs to hold in a small right neighbourhood of zero.

Similarly, in the second case we obtain $(h^*)^{-1}(s) = \sqrt{s}g^{-1}(\sqrt{s})$ with the same convexity requirement. Summarizing this discussion and neglecting constants c_0, c_1 we obtain:

Corollary 8.1. *If we assume that $g'(0) = 0$ (i.e. the damping is "weak"-superlinear at the origin) and the function $\sqrt{s}g(\sqrt{s})$ is convex for $s \in [0, s_0]$, where s_0 can be arbitrarily small, the differential equation to be solved becomes*

$$S_t + \sqrt{S}g(\sqrt{S}) = 0, \qquad S(0) = E(0) = S_0,$$

and $E(t) \leqslant C(E(0))S(t)$. More specifically, by integrating differential equation we obtain with $G(S, S_0) \equiv \int_{\sqrt{S_0}}^{\sqrt{S}} \frac{1}{g(u)}\,du$, $S(t) = G^{-1}(-\frac{t}{2}, S_0)$.

Corollary 8.2. *If we assume that $g(s)$ decays to zero at zero slower than any linear function, i.e.*

$$\lim_{s\to 0} \frac{s}{g(s)} = 0,$$

and moreover the function $\sqrt{s}g^{-1}(\sqrt{s})$ is convex for $s \in [0, s_0]$, where s_0 can be arbitrarily small, the differential equation to be solved becomes

$$S_t + \sqrt{S}g^{-1}\big(\sqrt{S}\big) = 0, \qquad S(0) = E(0) = S_0,$$

and $E(t) \leqslant C(E(0))S(t)$. More specifically, by integrating differential equation we obtain with $G(S, S_0) \equiv \int_{\sqrt{S_0}}^{\sqrt{S}} \frac{1}{g^{-1}(u)}\,du$, $S(t) = G^{-1}(-\frac{t}{2}, S_0)$.

Remark 8.1. We note that the expression involving $\sqrt{x}g^{-1}(\sqrt{x})$ appears in Proposition 2.2 [1], where dissipative systems without the sources are treated. This expression is used in [1] in order to define a certain convex function $H(x)$ which, in turn, is used for computation of effective decay rates. The approach used in [1] is very different and based on weighted integral inequalities, rather than on reducing PDE to ODE—as advocated in [23] and other papers that followed (see [17]). In fact, as illustrated above, the differential equation in Corollary 8.2 follows instantly from the intrinsic construction of the concave function h^*—where the latter was introduced for the first time in [23]. It is however interesting to see, that different approaches, when applied to comparable dynamics, lead to the same quantity describing asymptotic behavior of the corresponding solutions.

We shall illustrate the procedure with several examples. For the sake of clarity we normalize the constants so that they do not appear in the expressions.

Example 1. We consider $g(s) = s^p$, $p > 1$, at the origin. Since the function $s^{(p+1)/2}$ is convex for $p \geqslant 1$ we will be solving

$$S_t + S^{\frac{p+1}{2}} = 0. \tag{8.2}$$

This equation can be integrated directly, of course. However, for sake of illustration of the general formula we find

$$G(s, S_0) = \int_{\sqrt{S_0}}^{\sqrt{s}} u^{-p}\,du = \frac{1}{1-p}\big[s^{\frac{-p+1}{2}} - S_0^{\frac{-p+1}{2}}\big].$$

From here $G^{-1}(t) = [S_0^{(-p+1)/2} - t(1-p)]^{2/(-p+1)}$. Thus

$$E(t) \leqslant C\big(E(0)\big)\big[E(0)^{\frac{-p+1}{2}} + t(p-1)\big]^{\frac{2}{-p+1}}.$$

Of course, the same decay rates could be obtained by direct integration of (8.2).

Example 2. We take $g(s) = s^3 e^{-1/s^2}$ for s at the origin. Since the function $s^2 e^{-1/s}$ is convex in the neighbourhood of the origin we solve

$$S_t + S^2 e^{-\frac{1}{S}} = 0. \tag{8.3}$$

In this case $G(S, S_0) = -1/2[e^{-1/S} - e^{-1/S_0}]$ and $G^{-1}(t, S_0) = [\ln(e^{1/S_0} - 2t)]^{-1}$. Hence

$$E(t) \leqslant C\big(E(0)\big)\big[\ln\big(e^{\frac{1}{E(0)}} + t\big)\big]^{-1},$$

which solution could be also obtained directly from integrating (8.3).

Example 3. We consider $g(s) = s|s|e^{-1/|s|}$ for s near zero. Since the function $s^{3/2} e^{-\frac{1}{\sqrt{s}}}$ is convex on $[0, s_0]$ for some small s_0 we are led to differential equation

$$S_t + S^{3/2} e^{-\frac{1}{\sqrt{S}}} = 0. \tag{8.4}$$

Function $G(S, S_0)$ is given by $G(S, S_0) = -[e^{\frac{1}{\sqrt{S}}} - e^{1/\sqrt{S_0}}]$. Hence $G^{-1}(t, S_0) = \frac{1}{\ln^2[e^{1/\sqrt{S_0}} - t]}$ and

$$E(t) \leqslant C\big(E(0)\big)\frac{1}{\ln^2[e^{\frac{1}{\sqrt{E(0)}}} + \frac{1}{2}t]}.$$

Example 4. We take $g(s) = |s|^{\theta-1}s$, $0 < \theta < 1$. In this case the analysis is identical to the case of Example 1 since $g^{-1}(s) = s^{1/\theta}$, $s > 0$ and $\frac{1}{\theta} > 1$. Thus the decay rates in that case become

$$E(t) \leqslant C\big(E(0)\big)\left[E(0)^{\frac{-1+\theta}{2\theta}} + t\frac{1-\theta}{\theta}\right]^{\frac{2\theta}{\theta-1}}.$$

9. Blow up of solutions—Proof of Theorem 2.5

We shall consider *two polynomial competing sources* (boundary and interior), and without assuming any (polynomial) structure on the damping we will investigate the blow up phenomenon for the following problem:

$$\begin{cases} u_{tt} - \Delta u = |u|^{p-1}u & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial u}{\partial \nu} + u + g(u_t) = |u|^{k-1}u & \text{on } \Gamma \times (0, +\infty), \\ u(x, 0) = u^0(x) \in H^1(\Omega); \quad u_t(x, 0) = u^1(x) \in L^2(\Omega), & x \in \Omega. \end{cases} \tag{9.1}$$

In order to satisfy the assumptions imposed in Theorem 2.2 (local in time) it is sufficient to assume that:

$$
\begin{aligned}
& 1 \leqslant p \leqslant \frac{n+2}{n-2}, \quad \text{for } n \geqslant 3 \qquad \text{and} \qquad p \geqslant 1 \quad \text{for } n = 1, 2, \\
& 1 \leqslant k < \frac{n}{n-2}, \quad \text{for } n \geqslant 3 \qquad \text{and} \qquad k \geqslant 1 \quad \text{for } n = 1, 2, \\
& q > \frac{k}{r-k}; \quad r := \frac{2(n-1)}{n-2}, \\
& g \text{ is monotone and continuous} \quad \text{and} \\
& m_q |s|^{q+1} \leqslant g(s)s \leqslant M_q |s|^{q+1}, \quad \text{for all } s \in \mathbb{R}.
\end{aligned} \tag{9.2}
$$

Our goal is to determine a set of initial data where solutions blow up in finite time. Let us define the following set:

$$
B \cup C = \big\{ (\lambda, E) \in [0, +\infty) \times \mathbb{R};\ F(\lambda) \leqslant E < d;\ \lambda > \lambda_0 \big\},
$$

according to Fig. 1 in Section 2. We will prove that:

If $(\|u^0\|_{H^1(\Omega)}, E(0)) \in B \cup C$, or, in other words, assuming that

$$
E(0) < d \quad \text{and} \quad \big\|u^0\big\|_{H^1(\Omega)} > \lambda_0, \tag{9.3}
$$

then weak solutions will blow up in finite time.

The next lemma will play an essential role when proving the blow up.

Lemma 9.1. *Under the hypothesis given in* (9.2) *and* (9.3) *the weak solution to problem* (9.1) *mentioned in Theorem* 2.2 *verifies*

$$
\big\|u(t)\big\|_{H^1(\Omega)} \geqslant \lambda_2, \quad \textit{for some } \lambda_2 > \lambda_0,\ \textit{and for all } t \in [0, T_M).
$$

Moreover, the following inequality holds

$$
\big\|u(t)\big\|_{p+1}^{p+1} + \big\|u(t)\big\|_{k+1,\Gamma}^{k+1} \geqslant \frac{B_\Omega^{p+1}}{p+1} \lambda_2^{p+1} + \frac{B_\Gamma^{k+1}}{k+1} \lambda_2^{k+1}, \quad \textit{for all } t \in [0, T_M).
$$

Proof. (i) We observe that from (6.6) we have

$$
E\big(u(t)\big) \geqslant F\big(\big\|u(t)\big\|_{H^1(\Omega)}\big), \quad \forall t \in [0, T_M). \tag{9.4}
$$

We have that F is increasing for $0 < \lambda < \lambda_0$, decreasing for $\lambda > \lambda_0$, $F(\lambda_0) = d$, $F(\lambda) \to -\infty$ as $\lambda \to +\infty$. Then, since $d > E(u(0)) \geqslant F(\|u(0)\|_{H^1(\Omega)}) \geqslant F(0) = 0$, there exist $\lambda_2' < \lambda_0 < \lambda_2$, which verify

$$
F(\lambda_2) = F\big(\lambda_2'\big) = E\big(u(0)\big). \tag{9.5}
$$

Considering that $E(t)$ is non-increasing, we have

$$E\big(u(t)\big) \leqslant E\big(u(0)\big), \quad \forall t \in [0, T_M). \tag{9.6}$$

From (9.4) and (9.5) we deduce,

$$F\big(\|u(0)\|_{H^1(\Omega)}\big) \leqslant E\big(u(0)\big) = F(\lambda_2). \tag{9.7}$$

Since $\|u(0)\|_{H^1(\Omega)}, \lambda_2 \in (\lambda_0, +\infty)$ and F is deceasing in this interval, from (9.7) one has

$$\|u(0)\|_{H^1(\Omega)} \geqslant \lambda_2. \tag{9.8}$$

In the sequel, we will prove that

$$\|u(t)\|_{H^1(\Omega)} \geqslant \lambda_2, \quad \forall t \in [0, T_M). \tag{9.9}$$

In fact we will argue by contradiction. Supposing that (9.9) does not hold, then, there exists $t^* \in (0, T_M)$ such that

$$\|u(t^*)\|_{H^1(\Omega)} < \lambda_2. \tag{9.10}$$

If $\|u(t^*)\|_{H^1(\Omega)} > \lambda_0$, then, from (9.4), (9.5) and (9.10), we have

$$E\big(u(t^*)\big) \geqslant F\big(\|u(t^*)\|_{H^1(\Omega)}\big) > F(\lambda_2) = E\big(u(0)\big),$$

which contradicts (9.6) and proves (9.9). Now, if $\|u(t^*)\|_{H^1(\Omega)} \leqslant \lambda_0$, we have, taking (9.8) into account, that there exists $\bar{\lambda}$ which verifies

$$\|u(t^*)\|_{H^1(\Omega)} \leqslant \lambda_0 < \bar{\lambda} < \lambda_2 \leqslant \|u(0)\|_{H^1(\Omega)}. \tag{9.11}$$

Consequently, from the continuity of $\|u(\cdot)\|_{H^1(\Omega)}$ there exists $\bar{t} \in (0, t^*)$ verifying

$$\|u(\bar{t})\|_{H^1(\Omega)} = \bar{\lambda}.$$

From this last identity and from (9.4), (9.5) and (9.11) we obtain

$$E\big(u(\bar{t})\big) \geqslant F\big(\|u(\bar{t})\|_{H^1(\Omega)}\big) = F(\bar{\lambda}) > F(\lambda_2) = E\big(u(0)\big),$$

which also contradicts (9.6) and proves (9.9).

On the other hand, from the identity of the energy, it holds that

$$\begin{aligned}
&\frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{2}\|u(t)\|_{H^1(\Omega)}^2 \\
&\quad \leqslant E\big(u(0)\big) + \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} + \frac{1}{k+1}\|u(t)\|_{k+1,\Gamma}^{k+1}, \quad \forall t \in [0, T_M),
\end{aligned} \tag{9.12}$$

which implies, from (9.5), (9.9) and by the definition of F, that

$$\begin{aligned}
&\frac{1}{p+1}\big\|u(t)\big\|_{p+1}^{p+1}+\frac{1}{k+1}\big\|u(t)\big\|_{k+1,\Gamma}^{k+1}\\
&\quad\geqslant\frac{1}{2}\big\|u(t)\big\|_{H^1(\Omega)}^2-E\big(u(0)\big)\geqslant\frac{1}{2}\lambda_2^2-F(\lambda_2)\\
&\quad=\frac{B_\Omega^{p+1}}{p+1}\lambda_2^{p+1}+\frac{B_\Gamma^{k+1}}{k+1}\lambda_2^{k+1}.
\end{aligned}$$

This completes the proof of Lemma 9.1. □

9.1. Proof of Theorem 2.5

It is enough to prove that no global solution in $[0,+\infty)$ can exist. In fact, the local existence of solutions together with the standard continuation principle yields the blow up of the linear energy $\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|u(t)\|_{H^1(\Omega)}^2$ which, in view of the inequality

$$\begin{aligned}
&\frac{1}{2}\big\|u'(t)\big\|_2^2+\frac{1}{2}\big\|u(t)\big\|_{H^1(\Omega)}^2\\
&\quad\leqslant E\big(u(0)\big)+\frac{1}{p+1}\big\|u(t)\big\|_{p+1}^{p+1}+\frac{1}{k+1}\big\|u(t)\big\|_{k+1,\Gamma}^{k+1},\quad \forall t\in[0,T_M),
\end{aligned}$$

implies that

$$\lim_{t\to t_{\max}^-}\big(\big\|u(t)\big\|_{p+1}^{p+1}+\big\|u(t)\big\|_{k+1,\Gamma}^{k+1}\big)=+\infty,$$

which proves the theorem. Then, we will assume, by contradiction, that weak solutions exist in the whole interval $[0,+\infty)$.

Let E_2 be a real number such that $E(0)<E_2<d$. Setting

$$\mathcal{H}(t):=E_2-E(t),\quad t>0, \tag{9.13}$$

we have

$$\mathcal{H}'(t)=-E'(t)\geqslant 0, \tag{9.14}$$

which implies that $\mathcal{H}$ is non-decreasing, and, consequently

$$\mathcal{H}(t)\geqslant\mathcal{H}_0:=E_2-E(0)>0,\quad \forall t>0. \tag{9.15}$$

Considering Lemma 9.1, we have that $\|u(t)\|_{H^1(\Omega)}^2\geqslant\lambda_2$, for some $\lambda_2>\lambda_0$ and for all $t\geqslant 0$. From this inequality, the definition of the energy and taking (9.13) into account, we deduce

$$\begin{aligned}
\mathcal{H}(t)&=E_2-\left[\frac{1}{2}\big\|u'(t)\big\|_2^2+\frac{1}{2}\big\|u(t)\big\|_{H^1(\Omega)}^2-\frac{1}{p+1}\big\|u(t)\big\|_{p+1}^{p+1}-\frac{1}{k+1}\big\|u(t)\big\|_{k+1,\Gamma}^{k+1}\right]\\
&\leqslant E_2-\frac{1}{2}\big\|u(t)\big\|_{H^1(\Omega)}^2+\frac{1}{p+1}\big\|u(t)\big\|_{p+1}^{p+1}+\frac{1}{k+1}\big\|u(t)\big\|_{k+1,\Gamma}^{k+1}
\end{aligned}$$

$$< d - \frac{1}{2}\lambda_2^2 + \frac{1}{p+1}\left\|u(t)\right\|_{p+1}^{p+1} + \frac{1}{q+1}\left\|u(t)\right\|_{k+1,\Gamma}^{k+1},$$

which implies, having in mind that $d = F(\lambda_0) = \frac{\lambda_0^2}{2} - \frac{\lambda_0^{p+1}}{p+1} - \frac{\lambda_0^{k+1}}{k+1}$, that

$$\begin{aligned}
\mathcal{H}(t) &< \frac{\lambda_0^2}{2} - \frac{\lambda_0^{p+1}}{p+1} - \frac{\lambda_0^{k+1}}{k+1} - \frac{\lambda_2^2}{2} + \frac{1}{p+1}\left\|u(t)\right\|_{p+1}^{p+1} + \frac{1}{k+1}\left\|u(t)\right\|_{k+1,\Gamma}^{k+1} \\
&\leqslant -\frac{\lambda_0^{p+1}}{p+1} - \frac{\lambda_0^{k+1}}{k+1} + \frac{1}{p+1}\left\|u(t)\right\|_{p+1}^{p+1} + \frac{1}{k+1}\left\|u(t)\right\|_{k+1,\Gamma}^{k+1} \\
&\leqslant \frac{1}{p+1}\left\|u(t)\right\|_{p+1}^{p+1} + \frac{1}{k+1}\left\|u(t)\right\|_{k+1,\Gamma}^{k+1},
\end{aligned} \tag{9.16}$$

for all $t \geqslant 0$.

Next, we will prove that there exist C_1, C_2 and C_3 positive constants such that the following inequality holds

$$\begin{aligned}
I_1 &:= \frac{d}{dt}\int_\Omega u'\,u\,dx \\
&\geqslant 2\left\|u'(t)\right\|_2^2 + C_1\left\|u(t)\right\|_{p+1}^{p+1} + C_2\left\|u(t)\right\|_{k+1,\Gamma}^{k+1} + C_3\left\|u(t)\right\|_{H^1(\Omega)}^2 \\
&\quad + 2\mathcal{H}(t) - \int_\Gamma g(u')u\,d\Gamma,
\end{aligned} \tag{9.17}$$

for all $t \geqslant 0$. For simplicity from now on we will omit the dependence on time in the notation. It is possible to show that for weak solutions the following identity holds

$$I_1 = \|u'\|_2^2 - \|u\|_{H^1(\Omega)}^2 + \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma}^{k+1} - \int_\Gamma g(u')u\,d\Gamma.$$

Adding and subtracting terms suitably and noting that $-E = \mathcal{H} - E_2$ we arrive at

$$\begin{aligned}
I_1 = 2\|u'\|_2^2 &+ \left[1 - \frac{\theta}{p+1}\right]\|u\|_{p+1}^{p+1} + \left[\frac{\theta}{2} - 1\right]\|u\|_{H^1(\Omega)}^2 \\
&- \theta E_2 + \theta\mathcal{H} + \left[\frac{\theta}{2} - 1\right]\|u'\|_2^2 - \int_{\Gamma_1} g(u')u\,d\Gamma \\
&+ \left[1 - \frac{\theta}{k+1}\right]\|u\|_{k+1,\Gamma}^{k+1},
\end{aligned} \tag{9.18}$$

where θ is a positive constant.

We have two cases to consider:

- If $p > k$ take $\theta = \theta_\varepsilon = p + 1 - \varepsilon$ with $0 < p - k < \varepsilon < p - 1$.
- If $k > p$ take $\theta = \theta_\varepsilon = k + 1 - \varepsilon$ with $0 < k - p < \varepsilon < k - 1$.

From the fact that $\|u(t)\|^2_{H^1(\Omega)} \geqslant \lambda_2^2$ for some $\lambda_2 > \lambda_0$ and for all $t \geqslant 0$ (see Lemma 9.1), it holds that

$$\begin{aligned}
&\left[\frac{\theta_\varepsilon}{2} - 1\right]\|u\|^2_{H^1(\Omega)} - \theta_\varepsilon E_2 \\
&\quad = \left[\frac{\theta_\varepsilon}{2} - 1\right]\|u\|^2_{H^1(\Omega)} - \left[\frac{\theta_\varepsilon}{2} - 1\right]\frac{\lambda_0^2}{\lambda_2^2}\|u\|^2_{H^1(\Omega)} \\
&\qquad + \left[\frac{\theta_\varepsilon}{2} - 1\right]\frac{\lambda_0^2}{\lambda_2^2}\|u\|^2_{H^1(\Omega)} - \theta_\varepsilon E_2 \\
&\quad \geqslant \left[\frac{\theta_\varepsilon}{2} - 1\right]\left[1 - \frac{\lambda_0^2}{\lambda_2^2}\right]\|u\|^2_{H^1(\Omega)} + \left[\frac{\theta_\varepsilon}{2} - 1\right]\lambda_0^2 - \theta_\varepsilon E_2 \\
&\quad = \underbrace{\left[\frac{\theta_\varepsilon}{2} - 1\right]\left[1 - \frac{\lambda_0^2}{\lambda_2^2}\right]}_{:=K_1(\varepsilon)>0}\|u\|^2_{H^1(\Omega)} + \underbrace{\left[\frac{\theta_\varepsilon}{2} - 1\right]\lambda_0^2 - \theta_\varepsilon E_2}_{:=K_2(\varepsilon)}.
\end{aligned} \tag{9.19}$$

We observe that

$$K_2(\varepsilon) = \left(E_2 - \frac{\lambda_0^2}{2}\right)\varepsilon + \frac{(p-1)}{2}\lambda_0^2 - (p+1)E_2, \quad \text{if } p > k,$$
$$K_2(\varepsilon) = \left(E_2 - \frac{\lambda_0^2}{2}\right)\varepsilon + \frac{(k-1)}{2}\lambda_0^2 - (k+1)E_2, \quad \text{if } k > p.$$

We would like to prove that $K_2(\varepsilon)$ is also positive. We have,

$$\begin{aligned}
\frac{\lambda_0^2}{2} - E_2 &> \frac{\lambda_0^2}{2} - d \\
&> \frac{\lambda_0^2}{2} - \left[\frac{\lambda_0^2}{2} - \frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p+1} - \frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k+1}\right] > 0,
\end{aligned}$$

which implies that $(-\frac{\lambda_0^2}{2} + E_2) < 0$. Since

$$\begin{aligned}
\frac{p-1}{2}\lambda_0^2 - (p+1)E_2 &> \frac{p-1}{2}\lambda_0^2 - (p+1)d \\
&= \frac{p-1}{2}\lambda_0^2 - (p+1)\left[\frac{\lambda_0^2}{2} - \frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p+1} - \frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k+1}\right] \\
&= \lambda_0^2\left[-1 + \frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p-1} + \frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k-1}\right] = 0 \quad (\text{if } p > k)
\end{aligned}$$

and

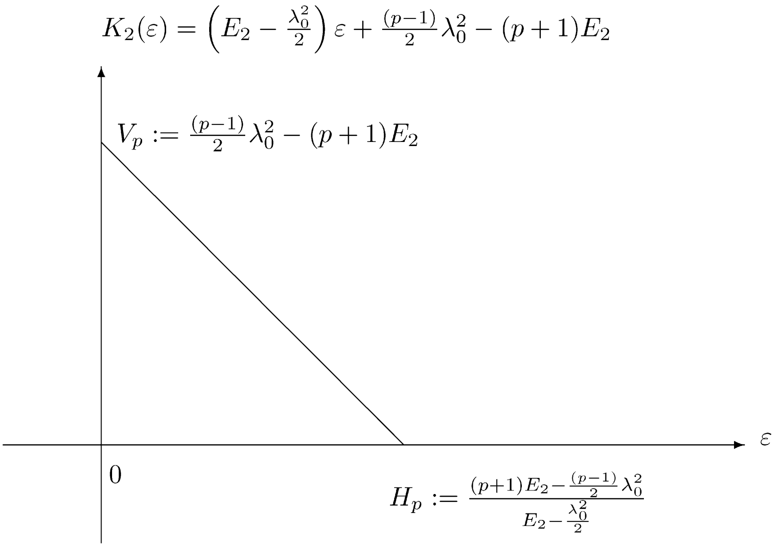

Fig. 3. The figure represents $K_2(\varepsilon)$ when $p > k$.

$$
\begin{aligned}
\frac{k-1}{2}\lambda_0^2 - (k+1)E_2 &> \frac{k-1}{2}\lambda_0^2 - (k+1)d \\
&= \frac{k-1}{2}\lambda_0^2 - (k+1)\left[\frac{\lambda_0^2}{2} - \frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p+1} - \frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k+1}\right] \\
&= \lambda_0^2\left[-1 + \frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p-1} + \frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k-1}\right] = 0 \quad (\text{if } k > p),
\end{aligned}
$$

$K_2(\varepsilon)$ represents a decreasing line which intercepts, respectively, the vertical and horizontal axes in the points $V_p := \frac{p-1}{2}\lambda_0^2 - (p+1)E_2$ if $p > k$ (or $V_q := \frac{k-1}{2}\lambda_0^2 - (k+1)E_2$ if $k > p$) and $H_p := \frac{(p-1)\lambda_0^2 - 2(p+1)E_2}{\lambda_0^2 - 2E_2}$ if $p > k$ (or $H_k := \frac{(k-1)\lambda_0^2 - 2(k+1)E_2}{\lambda_0^2 - 2E_2}$ if $k > p$). So,

$$
K_2(\varepsilon) > 0 \quad \text{if only if} \quad 0 < \varepsilon < H_p \text{ (respectively } 0 < \varepsilon < H_q). \tag{9.20}
$$

See Figs. 3 and 4.

Having the identity (9.18) in mind, we would like to choose ε small enough in order to have

$$
\left[1 - \frac{p+1-\varepsilon}{k+1}\right] > 0 \quad \text{or} \quad \left[1 - \frac{k+1-\varepsilon}{p+1}\right] > 0.
$$

But this implies that $\varepsilon > p - k$ (respectively $\varepsilon > k - p$), and, consequently, we have to consider $p > k$ (respectively $k > p$). But, since we have already considered $\varepsilon < p - 1$ (respectively $\varepsilon < k - 1$) we deduce that

$$
p - k < \varepsilon < p - 1 \quad (\text{respectively } k - p < \varepsilon < k - 1) \tag{9.21}
$$

which implies that we must have $p - k < p - 1$ (respectively $k - p < k - 1$), or still, $k > 1$ (respectively $p > 1$), what in fact we have.

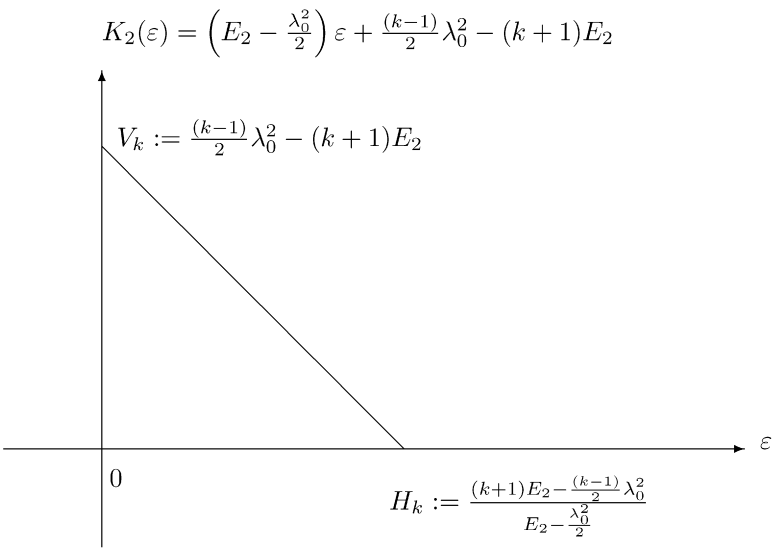

Fig. 4. The figure represents $K_2(\varepsilon)$ when $k > p$.

From (9.20) and (9.21) we would like to find ε such that $p-k < \varepsilon < H_p$ (respectively $k-p < \varepsilon < H_k$).

We observe that if $E(0) < 0$, then, we can choose $E(0) < E_2 < 0 < d$ so that we have:

$$E_2 < 0 \quad \Leftrightarrow \quad p-1 < H_p,$$
$$E_2 < 0 \quad \Leftrightarrow \quad k-1 < H_k.$$

So, when $E(0) < 0$ we have

$$p-k < \varepsilon < p-1 < H_p, \quad \text{if } p > k,$$
$$k-p < \varepsilon < k-1 < H_k, \quad \text{if } k > p,$$

and, consequently, taking ε small enough, from (9.18) and (9.19) we conclude the desired in (9.17).

Before analyzing the case $0 \leqslant E(0) < d$, let us firstly investigate when $p-k < H_p$, or in other words, when

$$(p-k)\big(\lambda_0^2 - 2E_2\big) < (p-1)\lambda_0^2 - 2(p+1)E_2. \tag{9.22}$$

But, the inequality (9.22) is equivalent to the following one

$$E_2 < \frac{\lambda_0^2(k-1)}{2(k+1)}. \tag{9.23}$$

So, we have two possibilities:

(i) $E(0) < \frac{\lambda_0^2(k-1)}{2(k+1)}$.
(ii) $E(0) \geqslant \frac{\lambda_0^2(k-1)}{2(k+1)}$.

In the first case above it is sufficient to investigate if $\frac{\lambda_0^2(k-1)}{2(k+1)} < d$. But this is true. In fact, having in mind that

$$d = \frac{\lambda_0^2}{2} - \frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p+1} - \frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k+1},$$

to prove the last inequality is the same as to prove that

$$\frac{(k-1)}{2(k+1)} < \frac{1}{2} - \frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p-1} - \frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k-1},$$

or still, that the following inequality holds

$$\frac{1}{k+1} > \frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p+1} + \frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k+1}. \tag{9.24}$$

In fact, since we are considering $p > k$ we deduce that

$$\frac{k+1}{p+1}B_\Omega\lambda_0^{p-1} + B_\Gamma^{k+1}\lambda_0^{k-1} < B_\Omega\lambda_0^{p-1} + B_\Gamma^{k+1}\lambda_0^{k-1} = 1.$$

From the last inequality we deduce (9.24), and, consequently, we can conclude that when $0 \leqslant E(0) < \frac{\lambda_0^2(k-1)}{2(k+1)}$ we can choose E_2 such that

$$0 < E(0) < E_2 < \frac{\lambda_0^2(k-1)}{2(k+1)} < d.$$

Moreover, from this last inequality and taking (9.23) into account, we can choose ε such that

$$p-k < \varepsilon < \min\left\{p-1, H_p := \frac{(p-1)\lambda_0^2 - 2(p+1)E_2}{\lambda_0^2 - 2E_2}\right\} = H_p. \tag{9.25}$$

Finally, when $E(0) \geqslant \frac{\lambda_0^2(k-1)}{2(k+1)}$, it seems NOT to be possible to find ε verifying the inequality (9.25). So, in this case we are forced to get the difference $p-k$ sufficiently small. Since the case $k-p < H_k$ is analogous, its analysis will be omitted.

Under the above considerations, returning to (9.19) and having in mind that

$$\lim_{\varepsilon\to 0}\theta_\varepsilon := \theta^*,$$

we deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0} K_2(\varepsilon) &= \left[\frac{\theta^*}{2}-1\right]\lambda_0^2 - \theta^* E_2\\
&> \left[\frac{\theta^*}{2}-1\right]\lambda_0^2 - \theta^* d\\
&= \underbrace{\left[\frac{\theta^*}{2}-1\right]\lambda_0^2 - \theta^*\left[\frac{\lambda_0^2}{2} - \frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p+1} - \frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k+1}\right].}_{:=J>0???}\end{aligned}$$

Next, we will prove that $J > 0$. Indeed, note that

$$J = \lambda_0^2\left[-1 + \theta^*\frac{B_\Omega^{p+1}}{p+1}\lambda_0^{p-1} + \theta^*\frac{B_\Gamma^{k+1}}{k+1}\lambda_0^{k-1}\right]. \tag{9.26}$$

Since

$$\theta^* = p+1 \quad \text{if } p > k \qquad \text{or} \qquad \theta^* = k+1 \quad \text{if } k > p,$$

we have, in view of (9.26), that

$$J > \lambda_0^2\big[\underbrace{-1 + B_\Omega^{p+1}\lambda_0^{p-1} + B_\Gamma^{k+1}\lambda_0^{k-1}}_{=0}\big] = 0,$$

which proves the desired in (9.17), that is, there exists a positive constants C_i, $i = 1, 2, 3$, such that the following inequality holds

$$\begin{aligned}I_1 &:= \frac{d}{dt}\int_\Omega u'u\,dx\\
&\geqslant 2\|u'(t)\|_2^2 + C_1\|u(t)\|_{p+1}^{p+1} + C_2\|u(t)\|_{k+1,\Gamma}^{k+1} + C_3\|u(t)\|_{H^1(\Omega)}^2\\
&\quad + 2\mathcal{H}(t) - \int_\Gamma g(u')u\,d\Gamma.\end{aligned} \tag{9.27}$$

Since $k > q$ we can choose $0 < \chi < \frac{k-q}{(q+1)(k+1)}$. With this choice, from the growth condition imposed on g, see (9.2), and noting that $L^{k+1}(\Gamma) \hookrightarrow L^{q+1}(\Gamma)$ we deduce

$$\begin{aligned}I_2 &= \left|\int_\Gamma g(u')u\,d\Gamma\right|\\
&\leqslant \|u'\|_{q+1,\Gamma}^q\,\|u\|_{q+1,\Gamma}\\
&\leqslant C\|u'\|_{q+1,\Gamma}^q\|u\|_{k+1,\Gamma}\big[\|u\|_{k+1}^{k+1} + \|u\|_{p+1}^{p+1}\big]^\chi\big[\|u\|_{k+1}^{k+1} + \|u\|_{p+1}^{p+1}\big]^{-\chi}\\
&\leqslant C\|u'\|_{q+1,\Gamma}^q\big[\|u\|_{k+1,\Gamma} + \|u\|_{p+1}^{p+1}\big]^{\frac{1}{k+1}}\big[\|u\|_{k+1}^{k+1} + \|u\|_{p+1}^{p+1}\big]^\chi\big[\|u\|_{k+1}^{k+1} + \|u\|_{p+1}^{p+1}\big]^{-\chi}\end{aligned}$$

$$\leqslant \big[C(\delta)\|u'\|_{q+1,\Gamma}^{q+1} + \delta\big\{\|u\|_{k+1,\Gamma}^{k+1} + \|u\|_{p+1}^{p+1}\big\}^{[\chi+\frac{1}{k+1}](q+1)}\big]\big[\|u\|_{k+1,\Gamma}^{k+1} + \|u\|_{p+1}^{p+1}\big]^{-\chi} \tag{9.28}$$

for some arbitrary positive δ.

On the other hand, since

$$\chi < \frac{k-q}{(q+1)(k+1)} \quad \Leftrightarrow \quad \left[\chi + \frac{1}{k+1}\right](q+1) < 1,$$

it holds that

$$\big\{\|u\|_{k+1,\Gamma}^{k+1} + \|u\|_{p+1}^{p+1}\big\}^{[\chi+\frac{1}{k+1}](q+1)} \leqslant \begin{cases} \leqslant \|u\|_{k+1,\Gamma}^{k+1} + \|u\|_{p+1}^{p+1}, & \text{if } \|u\|_{k+1,\Gamma}^{k+1} + \|u\|_{p+1}^{p+1} > 1, \\ \leqslant \mathcal{H}_0^{-1}\mathcal{H}_0, & \text{if } \|u\|_{k+1,\Gamma}^{k+1} + \|u\|_{p+1}^{p+1} \leqslant 1. \end{cases}$$

Recalling the inequality

$$\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma}^{k+1} \geqslant \mathcal{H}(t) \geqslant \mathcal{H}_0 > 0,$$

we deduce that

$$\big\{\|u\|_{k+1,\Gamma}^{k+1} + \|u\|_{p+1}^{p+1}\big\}^{[\chi+\frac{1}{k+1}](q+1)} \leqslant C\big(\mathcal{H}_0^{-1}\big)\big[\|u\|_{k+1,\Gamma}^{k+1} + \|u\|_{p+1}^{p+1}\big]$$

and, consequently, from (9.28)

$$\begin{aligned} I_2 &\leqslant C(\delta)\|u'\|_{q+1,\Gamma}^{q+1} \\ &\quad + \delta C\big(\mathcal{H}_0^{-1}\big)\big[\|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma}^{k+1}\big]\big[\|u\|_{k+1,\Gamma}^{k+1} + \|u\|_{p+1}^{p+1}\big]^{-\chi}. \end{aligned} \tag{9.29}$$

From (9.16) we have $\mathcal{H} \leqslant \|u\|_{k+1,\Gamma}^{k+1} + \|u\|_{p+1}^{p+1}$ which implies that

$$\big(\mathcal{H}(t)\big)^{-\chi} \geqslant \big[\|u\|_{k+1,\Gamma}^{k+1} + \|u\|_{p+1}^{p+1}\big]^{-\chi}. \tag{9.30}$$

Setting

$$\bar{\alpha} := \chi > 0,$$

it results from (9.14), (9.29), (9.30) and considering the identity of energy that

$$\begin{aligned} I_2 &\leqslant C_4\big\{C(\delta)\|u'\|_{q+1,\Gamma}^{q+1} + \delta\|u\|_{p+1}^{p+1} + \delta\|u\|_{k+1,\Gamma}^{k+1} + \delta\|u\|_{H^1(\Omega)}^2\big\}\mathcal{H}^{-\bar{\alpha}} \\ &\leqslant C_5\big\{C(\delta)\mathcal{H}' + \delta\|u\|_{k+1,\Gamma}^{k+1} + \delta\|u\|_{p+1}^{p+1} + \delta\|u\|_{H^1(\Omega)}^2\big\}\mathcal{H}^{-\bar{\alpha}}. \end{aligned} \tag{9.31}$$

Let us consider $0 < \alpha < \bar{\alpha}$. From (9.15), we have $\mathcal{H}(t) \geqslant \mathcal{H}_0 > 0$ for all $t \geqslant 0$ which leads to $\mathcal{H}^{\alpha-\bar{\alpha}} \leqslant \mathcal{H}_0^{\alpha-\bar{\alpha}}$. From this last inequality and taking (9.31) into account, we deduce

$$I_2 \leqslant C_6\big\{C(\delta)\mathcal{H}'\mathcal{H}_0^{\alpha-\bar{\alpha}}\mathcal{H}^{-\alpha} + \delta\|u\|_{k+1,\Gamma}^{k+1} + \delta\|u\|_{p+1}^{p+1} + \delta\|u\|_{H^1(\Omega)}^2\big\}, \tag{9.32}$$

where $C_6 = C_6(q, \bar{\alpha}, \Omega, \mathcal{H}_0)$. Combining (9.17) and (9.32) and choosing δ small enough we deduce that

$$I_1 \geqslant 2\|u'(t)\|_2^2 + \frac{C_1}{2}\|u\|_{p+1}^{p+1} + \frac{C_2}{2}\|u(t)\|_{k+1,\Gamma}^{k+1} + \frac{C_3}{2}\|u(t)\|_{H^1(\Omega)}^2 + 2\mathcal{H} - C_7\mathcal{H}'\mathcal{H}_0^{\alpha-\bar{\alpha}}\mathcal{H}^{-\alpha}. \tag{9.33}$$

Defining

$$\mathcal{F}(t) := \|u(t)\|_2^2 \quad \text{and} \quad \mathcal{Z}(t) := \mathcal{H}^{1-\alpha}(t) + \varepsilon\mathcal{F}'(t),$$

and considering the identity

$$\mathcal{Z}'(t) = \frac{d}{dt}\big(\mathcal{H}^{1-\alpha}(t) + \varepsilon\mathcal{F}'(t)\big) = (1-\alpha)\mathcal{H}^{-\alpha}(t)\mathcal{H}'(t) + \varepsilon\mathcal{F}''(t),$$

from (9.33) we deduce that

$$\begin{aligned}\mathcal{Z}'(t) &= \frac{d}{dt}\big(\mathcal{H}^{1-\alpha}(t) + \varepsilon\mathcal{F}'(t)\big) = (1-\alpha_s)\mathcal{H}^{-\alpha}(t)\mathcal{H}'(t) + \varepsilon\mathcal{F}''(t)\\ &\geqslant \big(1-\alpha-\varepsilon C_7\mathcal{H}_0^{\alpha-\bar{\alpha}}\big)\mathcal{H}'\mathcal{H}^{-\alpha}\\ &\quad + 2\varepsilon\|u'(t)\|_2^2 + \varepsilon\frac{C_1}{2}\|u(t)\|_{p+1}^{p+1} + \varepsilon\frac{C_2}{2}\|u(t)\|_{k+1,\Gamma}^{k+1} + \varepsilon\frac{C_3}{2}\|u(t)\|_{H^1(\Omega)}^2\\ &\quad + 2\varepsilon\mathcal{H}.\end{aligned}$$

Choosing $0 < \alpha < \min\{1, \bar{\alpha}\}$, and considering ε small enough we infer

$$\begin{aligned}\mathcal{Z}'(t) &= \frac{d}{dt}\big(\mathcal{H}^{1-\alpha}(t) + \varepsilon\mathcal{F}'(t)\big)\\ &\geqslant \varepsilon C\big(\mathcal{H} + \|u'(t)\|^2 + \|u(t)\|_{p+1}^{p+1} + \|u(t)\|_{k+1,\Gamma}^{k+1}\big),\end{aligned} \tag{9.34}$$

where C is a positive constant independent of ε. In particular, (9.34) shows that $\mathcal{Z}(t)$ is an increasing function with

$$\mathcal{Z}(t) = \mathcal{H}^{1-\alpha}(t) + \varepsilon\mathcal{F}'(t) \geqslant \mathcal{H}^{1-\alpha}(0) + \varepsilon\mathcal{F}'(0).$$

If $\mathcal{F}'(0) \geqslant 0$, then no further condition on ε is needed in order to have $\mathcal{H}^{1-\alpha}(0) + \varepsilon\mathcal{F}'(0) = \mathcal{Z}(0) > 0$. However, if $\mathcal{F}'(0) < 0$ letting ε sufficiently small, we also have $\mathcal{Z}(0) > 0$, so that, from (9.34) we conclude that

$$\mathcal{Z}(t) \geqslant \mathcal{Z}(0) > 0, \quad \forall t \geqslant 0. \tag{9.35}$$

Next, we will prove the estimate

$$\mathcal{Z}'(t) \geqslant \varepsilon C\mathcal{Z}^{\beta}(t), \quad \forall t \geqslant 0, \tag{9.36}$$

where C is a positive constant and $\beta = \frac{1}{1-\alpha} > 1$. (Note that $0 < \alpha < 1$.) In order to verify (9.36) we consider two cases:

Case A. $\mathcal{F}' \leqslant 0$. We have

$$\mathcal{Z}^\beta(t) = \left[\mathcal{H}^{1-\alpha}(t) + \varepsilon\mathcal{F}'(t)\right]^{\frac{1}{1-\alpha}} \leqslant \mathcal{H}(t),$$

so that from (9.16) and (9.34) we deduce that

$$\begin{aligned}\mathcal{Z}'(t) &\geqslant \varepsilon C\left(\mathcal{H} + \left\|u'(t)\right\|^2 + \left\|u(t)\right\|_{p+1}^{p+1} + \left\|u(t)\right\|_{k+1,\Gamma}^{k+1}\right)\\ &\geqslant \varepsilon C'\left(\mathcal{H}(t) + \left\|u'(t)\right\|^2\right)\\ &\geqslant \varepsilon C'\left(\mathcal{Z}^\beta(t) + \left\|u'(t)\right\|^2\right)\end{aligned}$$

which proves (9.36).

Case B. $\mathcal{F}' > 0$. We proceed as in Todorova and Georgiev [12, p. 307].
Having (9.34) in mind, (9.36) will be valid if the following inequality holds

$$\mathcal{H} + \|u'\|_2^2 + \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma}^{k+1} \geqslant C\big[\underbrace{\mathcal{H}^{1-\alpha} + \varepsilon\mathcal{F}'}_{=\mathcal{Z}(t)}\big]^\beta. \tag{9.37}$$

Indeed, we have for $0 < \varepsilon < 1$ that

$$\begin{aligned}\mathcal{Z}^\beta(t) &= \left[\mathcal{H}^{1-\alpha} + \varepsilon\mathcal{F}'\right]^\beta\\ &\leqslant C(\beta)\left[\mathcal{H}(t) + \left[\mathcal{F}'(t)\right]^\beta\right].\end{aligned} \tag{9.38}$$

So, taking (9.37) and (9.38) into account, it remains to estimate $[\mathcal{F}'(t)]^\beta$ in terms of $\|u'\|_2^2 + \|u\|_{p+1}^{p+1} + \|u\|_{k+1,\Gamma}^{k+1}$. In fact, making use of Hölder's inequality and noting that $L^{p+1}(\Omega) \hookrightarrow L^2(\Omega)$ we deduce

$$\left[\mathcal{F}'(t)\right]^\beta \leqslant C\|u\|_{p+1}^\beta \|u'\|_2^\beta.$$

Now, considering $\alpha_s < 1/2$ and employing Young's inequality having in mind that $\frac{2-\beta}{2} + \frac{\beta}{2} = 1$ we obtain

$$\left[\mathcal{F}'(t)\right]^\beta \leqslant C\left[\|u\|_{p+1}^{\frac{2\beta}{2-\beta}} + \|u'\|_2^2\right]. \tag{9.39}$$

Noting that

$$\begin{aligned}&\frac{2\beta}{2-\beta} = \frac{2}{1-\alpha},\\ &\frac{2}{1-\alpha} \leqslant p+1 \quad \Leftrightarrow \quad \alpha \leqslant \frac{p-1}{2(p+1)} \quad \left(< \frac{1}{2}\right)\end{aligned}$$

and recalling

$$C\big[\|u\|_{p+1}^{p+1}+\|u\|_{k+1,\Gamma}^{k+1}\big]\geqslant \mathcal{H}(t)\geqslant \mathcal{H}_0>0,$$

we infer

$$\|u\|_{p+1}^{\frac{2\beta}{2-\beta}}\leqslant C\big[\|u\|_{p+1}^{p+1}+\|u\|_{k+1,\Gamma}^{k+1}\big]. \tag{9.40}$$

Thus, combining (9.38) and (9.40) we obtain

$$\begin{aligned}\mathcal{Z}^{\beta}(t)&\leqslant C\big(\|u\|_{p+1}^{p+1}+\|u'\|_2^2\big)\\&\leqslant \big(\|u\|_{k+1,\Gamma}^{k+1}+\|u\|_{p+1}^{p+1}+\|u'\|_2^2\big).\end{aligned}$$

Combining the above inequality with (9.34) we deduce the desired in (9.36). From (9.35) and (9.36) we obtain in the standard way the finite blow up of $\mathcal{Z}(t)$ as well as the blow up of u. This completes the proof. □

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